

SCHOTTKY GROUPS ACTING ON HOMOGENEOUS RATIONAL MANIFOLDS

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ABSTRACT. We systematically study Schottky group actions on homogeneous rational manifolds and find two new families besides those given by Nori's well-known construction. This yields new examples of non-Kähler compact complex manifolds having free fundamental groups. We then investigate their analytic and geometric invariants such as the Kodaira and algebraic dimension, the Picard group and the deformation theory, thus extending results due to Lárusson and to Seade and Verjovsky. As a byproduct, we find previously unknown examples of equivariant compactifications of $\mathrm{SL}(2, \mathbb{C})/\Gamma$ for Γ a discrete free loxodromic subgroup of $\mathrm{SL}(2, \mathbb{C})$.

1. INTRODUCTION

A classical Schottky group acting on the Riemann sphere \mathbb{P}_1 is given as follows. Choose $2r$ open discs $U_1, V_1, \dots, U_r, V_r \subset \mathbb{P}_1$ having pairwise disjoint closures as well as r loxodromic automorphisms $\gamma_1, \dots, \gamma_r$ of \mathbb{P}_1 satisfying $\gamma_j(U_j) = \mathbb{P}_1 \setminus \overline{V_j}$. The group $\Gamma \subset \mathrm{Aut}(\mathbb{P}_1)$ generated by $\gamma_1, \dots, \gamma_r$ is a free group of rank r acting freely and properly on the open subset $\mathcal{U}_\Gamma := \Gamma \cdot \mathcal{F}_\Gamma$ where

$$\mathcal{F}_\Gamma := \mathbb{P}_1 \setminus \bigcup_{j=1}^r (U_j \cup V_j).$$

Moreover, the quotient $\mathcal{U}_\Gamma/\Gamma$ is a compact Riemann surface of genus r . One can relax the notion of a classical Schottky group by considering $2r$ pairwise disjoint open subsets of \mathbb{P}_1 that are bounded by arbitrary Jordan curves instead of circles. In this case Koebe showed that every compact Riemann surface can be obtained as quotient of an open subset of \mathbb{P}_1 by a Schottky group. We refer the reader to [CNS13, Chapter 1.2.5] for an account on the history of Schottky groups.

In [Nor86] Nori extended the construction of Schottky groups to higher dimensions in order to obtain compact complex manifolds having free fundamental group of any rank. Let us recall his construction. Let $z, w \in \mathbb{C}^{n+1}$ and consider the smooth function on \mathbb{P}_{2n+1} given by $\varphi[z : w] = \frac{\|w\|^2}{\|z\|^2 + \|w\|^2}$. The fibers $C_a = \varphi^{-1}(a)$ for $a = 0, 1$ are isomorphic to \mathbb{P}_n . For $0 < \varepsilon < \frac{1}{2}$ we have the open neighborhoods $U_\varepsilon = \{\varphi < \varepsilon\}$ and $V_\varepsilon = \{\varphi > 1 - \varepsilon\}$ of C_0 and C_1 , respectively. For $\lambda \in \mathbb{C}^*$ define an automorphism of \mathbb{P}_{2n+1} by $g_\lambda[z : w] := [\lambda^{-1}z : \lambda w]$. A direct calculation shows that g_λ maps U_ε biholomorphically to $\mathbb{P}_{2n+1} \setminus \overline{V_\varepsilon}$ if $|\lambda|^2 = \frac{1-\varepsilon}{\varepsilon} > 1$. Now let f_2, \dots, f_r be $r \geq 2$ automorphisms such that $C_0, C_1, f_2(C_0), f_2(C_1), \dots, f_r(C_0), f_r(C_1)$ are pairwise disjoint and take $\varepsilon > 0$ sufficiently small such that $U_\varepsilon, V_\varepsilon, f_2(U_\varepsilon), f_2(V_\varepsilon), \dots, f_r(U_\varepsilon), f_r(V_\varepsilon)$ have pairwise disjoint closures. The automorphisms

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f_2, \dots, f_r exist since $\text{Aut}(\mathbb{P}_{2n+1})$ acts transitively on the set of disjoint pairs of linearly embedded \mathbb{P}_n 's. Fix $\lambda \in \mathbb{C}^*$ with $|\lambda|^2 = \frac{1-\varepsilon}{\varepsilon}$ and define r automorphisms of \mathbb{P}_{2n+1} by $\gamma_1 := g_\lambda$ and $\gamma_j := f_j \circ \gamma_1 \circ f_j^{-1}$ for $2 \leq j \leq r$. The group $\Gamma \subset \text{Aut}(\mathbb{P}_{2n+1})$ generated by $\gamma_1, \dots, \gamma_r$ is an example of a Schottky group acting on \mathbb{P}_{2n+1} . As in the one-dimensional case, there is the analogously defined open subset \mathcal{U}_Γ on which Γ acts freely and properly such that the quotient $Q_\Gamma := \mathcal{U}_\Gamma / \Gamma$ is a compact complex manifold. The quotient manifolds Q_Γ obtained by Nori's construction were studied in a more general framework by Lárússon in [Lár98]. He showed that, under a technical assumption on the generators of the Schottky group Γ which guarantees that the $4n$ -dimensional Hausdorff measure of $\mathbb{P}_{2n+1} \setminus \mathcal{U}_\Gamma$ is zero, the manifold Q_Γ has Kodaira dimension $-\infty$, is rationally connected, and is not Moishezon. For Schottky groups acting on \mathbb{P}_3 he proved furthermore that Q_Γ has algebraic dimension zero. In [SV03] Seade and Verjovsky proved for arbitrary n that Q_Γ is diffeomorphic to a smooth fiber bundle over \mathbb{P}_n with fiber the connected sum of $r - 1$ copies of $S^1 \times S^{2n+1}$ and, furthermore, they studied the deformation theory of Q_Γ .

So far the only known examples of Schottky transformation groups are discrete subgroups of the automorphism group of \mathbb{P}_{2n+1} . Under the hypothesis that the 2-dimensional Hausdorff measure of $\mathbb{P}_2 \setminus \mathcal{U}_\Gamma$ is zero, Lárússon proved that there do not exist Schottky groups acting on \mathbb{P}_2 . In [Can08] Cano generalized this result to \mathbb{P}_{2n} .

This leads naturally to the main purpose of the present paper, namely the construction of Schottky group actions on homogeneous rational manifolds different from \mathbb{P}_{2n+1} .

In order to state the results we have to introduce some terminology. A *Schottky pair* in a connected compact complex manifold X is a pair of disjoint connected compact complex submanifolds C_0 and C_1 such that there is a holomorphic \mathbb{C}^* -action on X that is free and proper on $X \setminus (C_0 \cup C_1)$ and has fixed point set $X^{\mathbb{C}^*} = C_0 \cup C_1$. The first ingredient for the construction of new Schottky groups acting on homogeneous rational manifolds is the following observation.

Proposition 3.2. *Let G be a connected semisimple complex Lie group, let Q be a parabolic subgroup of G , and let G_0 be a non-compact real form of G . If the minimal G_0 -orbit in the homogeneous rational manifold $X = G/Q$ is a real hypersurface, then X admits a Schottky pair.*

Its proof is based on [Akh77] and Matsuki duality. In fact, the Schottky pairs (C_0, C_1) in $X = G/Q$ given by Proposition 3.2 are the compact orbits of $K = K_0^{\mathbb{C}}$ where K_0 is a maximal compact subgroup of G_0 .

This proposition strongly demands to classify all triplets (G, G_0, Q) such that the minimal G_0 -orbit in $X = G/Q$ is a hypersurface. Since we did not find this classification, which is of independent interest, in the literature, it is carried out in an appendix of this paper. As a consequence, the homogeneous rational manifolds admitting Schottky pairs coming from a minimal hypersurface orbit are \mathbb{P}_{2n+1} , the Grassmannians $\text{Gr}_n(\mathbb{C}^{2n})$, the quadrics Q_{2n} and the Grassmannians $\text{IGr}_n(\mathbb{C}^{2n+1})$ of subspaces of \mathbb{C}^{2n+1} that are isotropic with respect to a non-degenerate quadratic form on \mathbb{C}^{2n+1} . We proceed to determine all the cases in which the Schottky pairs can be moved by automorphisms of X in order to actually produce Schottky groups. Our main result is the following

Theorem 4.2. *Let G be a connected semisimple complex Lie group, let Q be a parabolic subgroup of G , and let G_0 be a non-compact real form of G whose minimal orbit is a hypersurface in $X = G/Q$. The Schottky pairs giving rise to Schottky group actions on X of arbitrary rank r are precisely the ones on \mathbb{P}_{2n+1} , Q_{4n+2} and $\text{IGr}_n(\mathbb{C}^{2n+1})$.*

In addition, we construct Schottky groups acting on Q_{2n+1} and on certain singular subvarieties of \mathbb{P}_{2n+1} which are not directly related to minimal hypersurface orbits.

Associated with a Schottky group Γ acting on X we have the quotient manifold Q_Γ . We prove that the compact complex manifold Q_Γ is non-Kähler, rationally connected, and has Kodaira dimension $\text{kod } Q_\Gamma = -\infty$, see Proposition 6.1. Furthermore, we give a criterion for the algebraic dimension $a(Q_\Gamma)$ to be zero (cf. Theorem 6.2) and construct examples of Q_Γ having strictly positive algebraic dimension, see Examples 6.4, 6.6 and 6.7. Their algebraic reduction leads to previously unknown almost-homogeneous compact complex manifolds, namely equivariant compactifications of H/Γ where H is the Zariski closure of Γ in $\text{Aut}(X)$. In particular, we obtain equivariant compactifications of $\text{SL}(2, \mathbb{C})/\Gamma$ for every discrete free loxodromic subgroup $\Gamma \subset \text{SL}(2, \mathbb{C})$ (cf. Example 6.5), which, as we hope, will lead to new insight in the theory of almost homogeneous 3-folds. These examples show that the statements of [CNS13, Proposition 9.3.12] respectively [SV03, Proposition 3.5] as well as of [CNS13, Theorem 9.3.17] respectively [SV03, Theorem 3.10] cannot be true in general.

We also determine the Picard group of Q_Γ (cf. Theorem 6.9) and establish the dimension and smoothness of its Kuranishi space of versal deformations (cf. Theorem 6.12). We note that several of these results are new even in the case $X = \mathbb{P}_{2n+1}$. Others have been obtained by Lárusson as well as Seade and Verjovsky under conditions on the Hausdorff dimension of $X \setminus \mathcal{U}_\Gamma$, which allowed them to apply extension theorems for holomorphic and meromorphic functions due to Shiffman and for cohomology classes due to Harvey. Replacing Shiffman's and Harvey's techniques by results of Andreotti-Grauert, Scheja and Merker-Porten, we are able to remove these assumptions on the Hausdorff dimension of $X \setminus \mathcal{U}_\Gamma$.

Let us outline the structure of the paper. In Section 2 we review the basic facts about Schottky groups in a generality suitable for our purpose. Sections 3 and 4 contain the proofs of Proposition 3.2 and Theorem 4.2, respectively. In Section 5 we present the technical tools needed to determine various cohomology groups of the quotient manifolds Q_Γ . These are then applied in the final Section 6 in order to obtain analytic and geometric invariants of Q_Γ as well as their deformation theory. The classification of the triplets (G, G_0, Q) such that the minimal G_0 -orbit in $X = G/Q$ is a hypersurface is carried out in the appendix.

2. COMPLEX SCHOTTKY GROUPS

In this section we define Schottky group actions on a connected compact complex manifold X in a way that is suitable for the context of this paper.

2.1. Schottky pairs. Let X be a connected compact complex manifold of complex dimension d . A *Schottky pair* in X is given by a pair (C_0, C_1) of connected compact complex submanifolds of X and a holomorphic \mathbb{C}^* -action on X with fixed point set $X^{\mathbb{C}^*} = C_0 \cup C_1$ that is free and proper on $X \setminus (C_0 \cup C_1)$. This \mathbb{C}^* -action corresponds to a holomorphic homomorphism $\mathbb{C}^* \rightarrow \text{Aut}(X)$ denoted by $\lambda \mapsto g_\lambda$.

Remark 2.1. Since the \mathbb{C}^* -action on $\Omega := X \setminus (C_0 \cup C_1)$ is free and proper, we get the trivial smooth principal $\mathbb{R}^{>0}$ -bundle $\Omega/S^1 \rightarrow \Omega/\mathbb{C}^*$, i.e., differentiably one has $\Omega/S^1 \simeq (\Omega/\mathbb{C}^*) \times \mathbb{R}^{>0}$. Therefore we can define an S^1 -invariant smooth auxiliary function $\varphi: \Omega \rightarrow (0, 1)$ as the composition of the projection onto the second factor with the identification $\mathbb{R}^{>0} \rightarrow (0, 1)$, $t \mapsto \frac{t^2}{1+t^2}$. Since $X^{\mathbb{C}^*} = C_0 \cup C_1$, we may extend φ continuously to a function $\varphi: X \rightarrow [0, 1]$ such that $C_0 = \varphi^{-1}(0)$ and $C_1 = \varphi^{-1}(1)$. One verifies directly

$$(2.1) \quad \varphi(g_\lambda(x)) = \frac{|\lambda|^4 \varphi(x)}{1 + (|\lambda|^4 - 1)\varphi(x)} =: \lambda \cdot \varphi(x).$$

In particular, φ is a submersion on Ω .

Remark 2.2. Later on we will choose the function $\varphi: X \rightarrow [0, 1]$ in a special way in order to have properties analogous to Nori's construction mentioned in the introduction.

For $0 < \varepsilon < \frac{1}{2}$ we set $U_\varepsilon := \{\varphi < \varepsilon\}$. Note that the family of these open sets forms a neighborhood basis of C_0 . Similarly, the open sets $V_\varepsilon := \{\varphi > 1 - \varepsilon\}$ give a neighborhood basis of C_1 .

Lemma 2.3. *Suppose that X admits a Schottky pair. Then*

- (a) *the \mathbb{C}^* -action on X maps fibers of φ to fibers of φ ,*
- (b) *for every $x \in X \setminus (C_0 \cup C_1)$ we have $\lim_{\lambda \rightarrow 0} g_\lambda(x) \in C_0$ and $\lim_{\lambda \rightarrow \infty} g_\lambda(x) \in C_1$, and*
- (c) *if $0 < \varepsilon < 1/2$ and $|\lambda|^2 = \frac{1-\varepsilon}{\varepsilon}$, then $g_\lambda(U_\varepsilon) = U_{1-\varepsilon} = X \setminus \overline{V_\varepsilon}$.*

Proof. The first two statements follow directly from the equivariance condition (2.1).

To show the third one, we calculate as follows. For $a \in \mathbb{R}^{\geq 0}$ we have

$$\frac{\frac{(1-\varepsilon)^2}{\varepsilon^2}a}{1 + \left(\frac{(1-\varepsilon)^2}{\varepsilon^2} - 1\right)a} = \frac{(1-\varepsilon)^2a}{\varepsilon^2 + (1-2\varepsilon)a},$$

and this quantity is less than $1 - \varepsilon$ if and only if $a < \varepsilon$. This shows $g_\lambda(U_\varepsilon) \subset U_{1-\varepsilon}$. In order to prove $g_\lambda^{-1}(U_{1-\varepsilon}) \subset U_\varepsilon$, let $a \in [0, 1 - \varepsilon]$ and consider

$$\frac{\frac{\varepsilon^2}{(1-\varepsilon)^2}a}{1 + \left(\frac{\varepsilon^2}{(1-\varepsilon)^2} - 1\right)a} = \frac{\varepsilon^2 a}{(1-\varepsilon)^2 + (2\varepsilon - 1)a} < \frac{\varepsilon^2(1-\varepsilon)}{(1-\varepsilon)^2 + (2\varepsilon - 1)(1-\varepsilon)} = \varepsilon,$$

as was to be shown. □

Remark 2.4. Suppose that X admits a Schottky pair (C_0, C_1) . Often, there exists in addition a holomorphic involution $s: X \rightarrow X$ such that

- (1) $\varphi \circ s = 1 - \varphi$ and
- (2) $s \circ g_\lambda = g_{\lambda^{-1}} \circ s$ for all $\lambda \in \mathbb{C}^*$.

In this case $s(C_0) = C_1$, hence C_0 and C_1 are biholomorphic. Moreover, the hypersurface $H := \{\varphi = 1/2\}$ is s -stable. Since s yields a biholomorphism between $U_{1/2}$ and $V_{1/2}$, the hypersurface H must be Levi-symmetric.

2.2. Movable Schottky pairs and Schottky groups. Let X be a connected compact manifold with $\dim_{\mathbb{C}} X = d$ that admits a Schottky pair (C_0, C_1) . We say that this Schottky pair *can be moved* or is *movable* if for every integer $r \geq 2$ there exist automorphisms f_2, \dots, f_r of X such that $C_0, C_1, f_2(C_0), f_2(C_1), \dots, f_r(C_0), f_r(C_1)$ are pairwise disjoint.

Example 2.5. As shown in the introduction, Nori's construction produces movable Schottky pairs in $X = \mathbb{P}_{2n+1}$.

Example 2.6. While $X = \mathbb{P}_2$ contains many Schottky pairs, see Proposition 3.2 and Theorem A.1, none of them is movable. To see this, suppose on the contrary that (C_0, C_1) is a movable Schottky pair in \mathbb{P}_2 . Since any two curves in \mathbb{P}_2 intersect, C_0 and C_1 must be points. Choose $\varepsilon > 0$ sufficiently small so that U_ε is contained in a ball. Consequently, V_ε contains a domain biholomorphic to $\mathbb{P}_2 \setminus \mathbb{B}_2$. But this is impossible since such domains cannot form a neighborhood basis of a point. We refer the reader to [Can08] for a related observation.

Suppose that (C_0, C_1) is movable and fix $f_1, \dots, f_r \in \text{Aut}(X)$ as above where $f_1 := \text{id}_X$. For all $1 \leq j \leq r$ choose $\varepsilon_j \in (0, 1/2)$ and $\lambda_j \in \mathbb{C}^*$ with $|\lambda_j|^2 = \frac{1-\varepsilon_j}{\varepsilon_j} > 1$. Set $\gamma_j := f_j \circ g_{\lambda_j} \circ f_j^{-1}$ and $U_j := f_j(U_{\varepsilon_j})$ and $V_j := f_j(V_{\varepsilon_j})$. We always choose ε_j sufficiently small such that the open sets $U_1, \dots, U_r, V_1, \dots, V_r$ have pairwise disjoint closures.

The group $\Gamma \subset \text{Aut}(X)$ generated by $\gamma_1, \dots, \gamma_r$ is called a *Schottky group* associated with the movable Schottky pair (C_0, C_1) . For such a group Γ we define

$$\mathcal{F}_\Gamma := X \setminus \bigcup_{j=1}^r (U_j \cup V_j) \quad \text{and} \quad \mathcal{U}_\Gamma := \bigcup_{\gamma \in \Gamma} \gamma(\mathcal{F}_\Gamma).$$

It is clear that \mathcal{U}_Γ is a Γ -invariant domain in X .

Moreover, if X is simply-connected and if $\text{codim } C_0, \text{codim } C_1 \geq 2$, then \mathcal{U}_Γ is likewise simply-connected. This follows from the fact that \mathcal{U}_Γ is an increasing union of open subsets which are homotopy equivalent to $X \setminus C$ where C is the disjoint union of N copies of $C_0 \cup C_1$, see Subsection 6.2. If $\text{codim } C_0, \text{codim } C_1 \geq 2$, then each of these open sets is simply-connected, hence the same holds for \mathcal{U}_Γ .

The proof of [CNS13, Proposition 9.2.8] extends literally to give the following.

Proposition 2.7. *The Schottky group Γ is the free group generated by $\gamma_1, \dots, \gamma_r$ and acts freely and properly on \mathcal{U}_Γ . The connected set \mathcal{F}_Γ is a fundamental domain for the Γ -action on \mathcal{U}_Γ . Consequently the quotient $Q_\Gamma := \mathcal{U}_\Gamma / \Gamma$ is a connected compact complex manifold. If X is simply-connected and if $\text{codim } C_j \geq 2$ for $j = 0, 1$, then the fundamental group of Q_Γ is isomorphic to Γ .*

Remark 2.8. If we take $r = 1$, then we have $\Gamma \simeq \mathbb{Z}$ and $\mathcal{U}_\Gamma = X \setminus (C_0 \cup C_1) = \Omega$. In this case Q_Γ is a holomorphic fiber bundle over Ω / \mathbb{C}^* with an elliptic curve as fiber.

3. SCHOTTKY PAIRS ASSOCIATED WITH COMPACT HYPERSURFACE ORBITS

In this section we prove Proposition 3.2 which provides a general method to construct Schottky pairs in homogeneous rational manifolds.

3.1. Nori's construction. We start by reformulating Nori's construction of Schottky groups in group-theoretical terms. Recall that on $X = \mathbb{P}_{2n+1}$ we have the function

$$\varphi[z : w] := \frac{\|w\|^2}{\|z\|^2 + \|w\|^2},$$

where $(z, w) \in (\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}) \setminus \{0\}$. The hypersurface $H = \{\varphi = 1/2\} = \{\|z\|^2 - \|w\|^2 = 0\}$ is an orbit of the real form $G_0 := \text{SU}(n+1, n+1)$ of $G = \text{SL}(2n+2, \mathbb{C})$. Note that $X = \{\varphi < 1/2\} \cup H \cup \{\varphi > 1/2\}$ gives the decomposition of X into G_0 -orbits. Let K be the complexification of the maximal compact subgroup

$$K_0 := \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}; A, B \in \text{U}(n+1), \det(A) \det(B) = 1 \right\} \simeq \text{S}(\text{U}(n+1) \times \text{U}(n+1))$$

of G_0 . One sees directly that K has likewise precisely three orbits in X , namely the compact orbits $C_0 = \{[z : 0]\}$ and $C_1 = \{[0 : w]\}$, and the open orbit $\Omega = K \cdot H = X \setminus (C_0 \cup C_1)$. Moreover, for every $\lambda \in \mathbb{C}^*$ the automorphism $g_\lambda \in \text{Aut}(X)$ belongs to the center of K .

Remark 3.1. Note that the symplectic group $\tilde{G} := \text{Sp}(n+1, \mathbb{C}) \subset G$ acts transitively on $X = \mathbb{P}_{2n+1}$, too, see [Oni62] or [Ste82]. Moreover, the automorphism g_λ is contained in \tilde{G} for any $\lambda \in \mathbb{C}^*$. Since \tilde{G} has a Zariski-open orbit in $\text{Gr}_{n+1}(\mathbb{C}^{2n+1})$, we can construct Schottky groups acting on X also inside the symplectic group. In particular, the Zariski closure of such a Schottky group is contained in $\text{Sp}(n+1, \mathbb{C})$.

If $n+1 = 2k$, the hypersurface H is an orbit of the real form $\tilde{G}_0 := \text{Sp}(k, k)$ of $\text{Sp}(n+1, \mathbb{C})$ and $C_0, C_1 \simeq \mathbb{P}_{2k-1}$ are orbits of $\tilde{K} := \text{Sp}(k, \mathbb{C}) \times \text{Sp}(k, \mathbb{C})$. In other words, in this case we have again a real form having a compact hypersurface orbit.

This observation leads to a systematic way to construct Schottky group actions on homogeneous rational manifolds described in the next subsection.

3.2. Schottky pairs associated with compact hypersurface orbits. The following proposition allows to associate a Schottky pair with a compact hypersurface orbit of a real form G_0 of G acting on a homogeneous rational manifold $X = G/Q$. Its proof is based on Matsuki duality and on Akhiezer's paper [Akh77].

Proposition 3.2. *Let G be a connected complex semisimple group, let Q be a parabolic subgroup of G , and let $X = G/Q$ be the corresponding homogeneous rational manifold. Let G_0 be a non-compact real form of G such that the minimal G_0 -orbit in X is a real hypersurface. Then X admits a Schottky pair.*

Before giving the proof let us review the basic ideas of Matsuki duality. Let G_0 be a non-compact real form of G and let K_0 be a maximal compact subgroup of G_0 . We assume that the groups G_0 and K_0 are connected. We consider the complexification $K := K_0^{\mathbb{C}}$ as a subgroup of G and call it a *Matsuki partner* of G_0 . Matsuki duality provides a bijection between the G_0 -orbits and the K -orbits in $X = G/Q$, under which open G_0 -orbits correspond to compact K -orbits and compact G_0 -orbits to open K -orbits, see e.g. [BL02]. This implies in particular that G_0 has exactly one compact orbit in $X = G/Q$. This compact orbit has minimal dimension among all G_0 -orbits and will be called the *minimal G_0 -orbit* in X .

Suppose from now on that the compact G_0 -orbit in $X = G/Q$ is a hypersurface. In the appendix we will determine all triplets (G, Q, G_0) for which this is the case. First we shall deduce some information about the orbits of G_0 and K in X .

Lemma 3.3. *Suppose that the minimal G_0 -orbit in $X = G/Q$ is a hypersurface. Then X contains exactly three G_0 -orbits, the minimal one and two open ones. Moreover, the generic K_0 -orbit in X is a hypersurface as well.*

Proof. Since $X = G/Q$ is simply connected, the complement of the minimal G_0 -orbit has exactly two connected components by the Jordan-Brouwer separation theorem. The first claim follows from the fact that G_0 must act transitively on these connected components. For the second one, it is sufficient to note that K_0 acts transitively on the minimal G_0 -orbit. \square

Using Matsuki duality we see that the group K has likewise exactly three orbits in X : two compact ones which lie in the open G_0 -orbits and one open orbit that contains the compact G_0 -orbit. We denote the two compact K -orbits by C_0 and C_1 .

Proof of Proposition 3.2. We only have to prove the existence of a holomorphic \mathbb{C}^* -action on $X = G/Q$ that verifies the definition of a Schottky pair. Let $\Omega \simeq K/K_x$ be the open K -orbit in X . According to [Akh77, Theorem 1] its isotropy group is of the form $K_x = P_\chi$ where $P \subset K$ is a parabolic subgroup and P_χ denotes the kernel of a non-trivial character $\chi: P \rightarrow \mathbb{C}^*$ on P . In other words, the fibration $\Omega \simeq K/K_x \rightarrow K/P$ is a \mathbb{C}^* -principal bundle. Hence, there is a free and proper holomorphic \mathbb{C}^* -action on Ω .

It follows from [Akh77, Theorem 2] that this \mathbb{C}^* -action extends to all of X in such a way that the two compact K -orbits C_0 and C_1 are fixed pointwise. \square

4. HOMOGENEOUS RATIONAL MANIFOLDS ADMITTING MOVABLE SCHOTTKY PAIRS

Let $X = G/Q$ be a homogeneous rational manifold where G is a connected semisimple complex Lie group and let G_0 be a real form of G . In this section we discuss in detail all the examples of compact hypersurface orbits of G_0 that give rise to movable Schottky pairs. As shown in the appendix, the only cases where the minimal G_0 -orbit is a hypersurface in $X = G/Q$ are the following, see Theorem A.1.

- (1) $G_0 = \mathrm{SU}(p, q)$ acting on $X = \mathbb{P}_{p+q-1}$;
- (2) $G_0 = \mathrm{Sp}(p, q)$ acting on $X = \mathbb{P}_{2(p+q)-1}$;
- (3) $G_0 = \mathrm{SU}(1, n)$ acting on $X = \mathrm{Gr}_k(\mathbb{C}^{n+1})$;
- (4) $G_0 = \mathrm{SO}^*(2n)$ acting on $X = Q_{2n-2}$;
- (5) $G_0 = \mathrm{SO}(1, 2n)$ acting on $X = \mathrm{IGr}_n(\mathbb{C}^{2n+1})$;
- (6) $G_0 = \mathrm{SO}(2, 2n)$ acting on $X = \mathrm{IGr}_{n+1}(\mathbb{C}^{2n+2})^0$.

Here $\mathrm{IGr}_k(\mathbb{C}^n)$ is the set of all k -dimensional subspaces of \mathbb{C}^n which are isotropic with respect to a non-degenerate symmetric bilinear form. The homogeneous rational manifold $\mathrm{IGr}_k(\mathbb{C}^{2k})$ has two isomorphic connected components, see [GH78, Proposition, p. 735]. We denote by $\mathrm{IGr}_k(\mathbb{C}^{2k})^0$ one of these components.

Remark 4.1. It is well-known that there exists an $\mathrm{SO}(2n+1, \mathbb{C})$ -equivariant biholomorphism between $\mathrm{IGr}_n(\mathbb{C}^{2n+1})$ and $\mathrm{IGr}_{n+1}(\mathbb{C}^{2n+2})^0$. This corresponds to the fact that the automorphism group of $\mathrm{IGr}_n(\mathbb{C}^{2n+1})$ is isomorphic to $\mathrm{SO}(2n+2, \mathbb{C})$, see [Oni62] and [Ste82].

Important assumption. In all cases in which we obtain a Schottky group action associated with a compact hypersurface orbit as described in Proposition 3.2, we may and will choose the function φ introduced in Remark 2.1 to be K_0 -invariant, as it was done in Subsection 3.1 for $X = \mathbb{P}_{2n+1}$. This important assumption will assure the existence of subvarieties of \mathcal{U}_Γ , and therefore of Q_Γ , which are biholomorphic to the Schottky pair varieties C_0 and C_1 . This fact can be easily verified in each of the examples discussed in this section and will be crucial for several arguments in the proofs of complex analytic and geometric properties of the quotient varieties Q_Γ .

The main result of this section is

Theorem 4.2. *Let G be a connected semisimple complex Lie group, let Q be a parabolic subgroup of G , and let G_0 be a non-compact real form of G whose minimal orbit is a hypersurface in $X = G/Q$. The Schottky pairs giving rise to Schottky group actions on X of arbitrary rank r are precisely the ones on the odd-dimensional projective space \mathbb{P}_{2n+1} , the quadric Q_{4n+2} and the isotropic Grassmannian $X_n := \mathrm{IGr}_n(\mathbb{C}^{2n+1})$. Furthermore, if (C_0, C_1) denotes a Schottky pair, then $C_0 \simeq C_1$ is a linear \mathbb{P}_n in the case $X = \mathbb{P}_{2n+1}$, a linear \mathbb{P}_{2n+1} in the case $X = Q_{4n+2}$ and an equivariantly embedded copy of $X_{n-1} = \mathrm{IGr}_{n-1}(\mathbb{C}^{2n-1})$ in the case of X_n . In each of these three cases the automorphism group of X acts transitively on the set of Schottky pairs.*

The proof is given by considering separately all of the above six cases.

4.1. The case of projective space. The Schottky pairs coming from the first two entries in the above list are only movable if $p = q$: In both cases we have $C_0 \simeq \mathbb{P}_{p-1}$ and $C_1 \simeq \mathbb{P}_{q-1}$. If $p < q$, then $\dim C_1 \geq \frac{1}{2} \dim X$. Hence, C_1 cannot be moved away from itself unless $p = q$ in which case we get back Nori's construction, see Subsection 3.1.

It is not hard to see that $G = \mathrm{SL}(2n, \mathbb{C})$ acts transitively on the set of Schottky pairs in $X = \mathbb{P}_{2n-1}$, i.e., that the set

$$\{(C_0, C_1) \in \mathrm{Gr}_n(\mathbb{C}^{2n}) \times \mathrm{Gr}_n(\mathbb{C}^{2n}); C_0 \cap C_1 = \{0\}\}$$

is an $\mathrm{SL}(2n, \mathbb{C})$ -orbit with respect to the diagonal action on $\mathrm{Gr}_n(\mathbb{C}^{2n}) \times \mathrm{Gr}_n(\mathbb{C}^{2n})$.

4.2. The case of complex Grassmannians. Let us consider the action of $G_0 = \mathrm{SU}(1, n)$ on $X = \mathrm{Gr}_k(\mathbb{C}^{n+1})$ for $1 \leq k \leq n$. Here we have $K = \mathrm{GL}(n, \mathbb{C})$ and the K -action on X is induced from the K -representation on $\mathbb{C}^{n+1} = \mathbb{C}e_1 \oplus (\{0\} \times \mathbb{C}^n)$ where $e_1 = (1, 0, \dots, 0)$.

The compact K -orbits in X are

$$\begin{aligned} C_0 &= \{V \in X; V \subset \{z_1 = 0\}\} \simeq \text{Gr}_k(\mathbb{C}^n) \quad \text{and} \\ C_1 &= \{V \in X; e_1 \in V\} \simeq \text{Gr}_{k-1}(\mathbb{C}^n). \end{aligned}$$

We claim that C_0 can only be moved away by an automorphism of X if $k = n$. Indeed, suppose that $C_0 \cap f(C_0) = \emptyset$ for some $f \in \text{Aut}(X)$. Then $f(C_0)$ is the set of all k -dimensional subspaces of \mathbb{C}^{n+1} that are contained in a fixed hyperplane H of \mathbb{C}^{n+1} . Since $\dim((\{0\} \times \mathbb{C}^n) \cap H) \geq n - 1$, the subsets C_0 and $f(C_0)$ cannot be disjoint for $k \leq n - 1$. A similar argument shows that C_1 and $f(C_1)$ can only be disjoint for $k = 1$. Consequently, this Schottky pair in $X = \text{Gr}_k(\mathbb{C}^{n+1})$ is only movable for $k = n = 1$. In this case we obtain Schottky groups acting on \mathbb{P}_1 .

4.3. Schottky groups acting on Q_{2n-2} . Let us consider the symmetric bilinear form b on \mathbb{C}^{2n} given by the matrix $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ and let G be the group of its linear isometries having determinant 1. Then $G \simeq \text{SO}(2n, \mathbb{C})$ acts transitively on the even-dimensional quadric $Q_{2n-2} := \{[z : w] \in \mathbb{P}_{2n-1}; q(z, w) = 0\}$ where

$$q(z, w) = \langle z, w \rangle = z_1 w_1 + \cdots + z_n w_n$$

is the quadratic form associated with b .

Due to Theorem A.1 the real form $G_0 = \text{SO}^*(2n) = G \cap \text{SU}(n, n)$ has a compact hypersurface orbit in $X = Q_{2n-2}$. One verifies directly that the Lie algebra of G has the form

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}; A \in \mathbb{C}^{n \times n}, B, C \in \mathfrak{so}(n, \mathbb{C}) \right\}$$

and that a Matsuki partner of G_0 is given by $K = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}; A \in \text{GL}(n, \mathbb{C}) \right\}$. The two compact K -orbits in X are

$$C_0 = \{[z : 0]; z \in \mathbb{C}^n\} \simeq \mathbb{P}_{n-1} \quad \text{and} \quad C_1 = \{[0 : w]; w \in \mathbb{C}^n\} \simeq \mathbb{P}_{n-1},$$

and they form a Schottky pair. The function φ will always be chosen as

$$\varphi[z : w] := \frac{\|w\|^2}{\|z\|^2 + \|w\|^2}.$$

We claim that this Schottky pair is movable if and only if n is even. Suppose first that n is odd. Since G is connected, for every $f \in G$ the subvarieties C_0 and $f(C_0)$ belong to the same connected component of the set of $(n - 1)$ -planes in Q_{2n-2} which we identify with $\text{IGr}_n(\mathbb{C}^{2n})$. According to [GH78, Proposition, p. 735] this implies for their intersection in Q_{2n-2} that

$$\dim(C_0 \cap f(C_0)) \equiv n - 1 \pmod{2} = 0.$$

Thus $C_0 \cap f(C_0)$ is at least 0-dimensional, i.e., C_0 and $f(C_0)$ cannot be disjoint in X .

Remark 4.3. For $n = 3$ we have $Q_4 \simeq \text{Gr}_2(\mathbb{C}^4)$ where we have already seen that the Schottky pairs are not movable.

Now suppose that n is even. We will show that for a generic choice of $B, C \in \mathfrak{so}(n, \mathbb{C})$ the automorphism

$$f_{B,C} := f_B \circ f_C := \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ C & I_n \end{pmatrix} \in G$$

is such that $C_0, C_1, f_{B,C}(C_0)$ and $f_{B,C}(C_1)$ are pairwise disjoint. This follows essentially from the fact that for n even generic matrices in $\mathfrak{so}(n, \mathbb{C})$ are invertible. More precisely, note that for invertible $B, C \in \mathfrak{so}(n, \mathbb{C})$ the subspaces C_0, C_1 and $f_B(C_1)$ (resp. $C_0, C_1, f_C(C_0)$) are

pairwise disjoint. Since $f_{B,C}(C_0) = \{[(I_n + BC)z : Cz]; z \in \mathbb{P}_n\}$, the claim follows once we choose B and C invertible such that $I_n + BC$ is likewise invertible.

In conclusion, we obtain movable Schottky pairs and therefore Schottky group actions only on $X = Q_{4k-2}$. Note that in this case the Schottky pairs are given by Schottky pairs in \mathbb{P}_{4k-1} lying in Q_{4k-2} . Consequently, there exist Schottky groups acting on \mathbb{P}_{4k-1} that leave Q_{4k-2} invariant. This means that the quotient manifolds Q_Γ obtained from \mathbb{P}_{4k-1} contain a hypersurface.

Remark 4.4. Lárússon already observed the existence of Schottky groups acting on \mathbb{P}_3 leaving the quadric $Q_2 \simeq \mathbb{P}_1 \times \mathbb{P}_1$ invariant, see [Lár98, Proposition 2.2].

In closing we note that $G \simeq \mathrm{SO}(4k, \mathbb{C})$ acts transitively on the set of Schottky pairs in $X = Q_{4k-2}$, i.e., that the set

$$\{(C_0, C_1) \in \mathrm{IGr}_{2k}(\mathbb{C}^{4k})^0 \times \mathrm{IGr}_{2k}(\mathbb{C}^{4k})^0; C_0 \cap C_1 = \{0\}\}$$

is a G -orbit with respect to the diagonal action. To prove this, we will show that we can map any Schottky pair (C'_0, C'_1) to (C_0, C_1) by some element of G where $C_0 = \{[z : 0] \in X; z \in \mathbb{C}^{2k}\}$ and $C_1 = \{[0 : w] \in X; w \in \mathbb{C}^{2k}\}$. There exists $g \in G$ with $g(C'_0) = C_0$. Since $g(C'_1)$ is an isotropic subspace of \mathbb{C}^{4k} complementary to C_0 , it projects surjectively onto the subspace $\{z \in \mathbb{C}^{4k}; z_1 = \dots = z_{2k} = 0\}$. Therefore we find a basis of $g(C'_1)$ consisting of the vectors

$$(v_1, e_1), \dots, (v_{2k}, e_{2k})$$

where $v_i \in \mathbb{C}^{2k}$ and (e_1, \dots, e_{2k}) denotes the standard basis of \mathbb{C}^{2k} . The fact that $g(C'_1)$ is isotropic means that the matrix $B := (v_{ij}) \in \mathbb{C}^{2k \times 2k}$ is skew-symmetric. Hence, the element

$$g' := \begin{pmatrix} I_{2k} & B \\ 0 & I_{2k} \end{pmatrix} \in G$$

fixes C_0 and maps C_1 onto $g(C'_1)$, which concludes the argument.

4.4. Schottky groups acting on isotropic Grassmannians. Let $X_n = \mathrm{IGr}_n(\mathbb{C}^{2n+1})$ be the set of n -dimensional complex subspaces of \mathbb{C}^{2n+1} that are isotropic with respect to the quadratic form $q(u, z, w) = u^2 + 2\langle z, w \rangle$, where $u \in \mathbb{C}, z, w \in \mathbb{C}^n$. Then X_n is a homogeneous rational manifold of dimension $\dim_{\mathbb{C}} X_n = \frac{n(n+1)}{2}$. The connected isometry group $G \simeq \mathrm{SO}(2n+1, \mathbb{C})$ of q acts transitively on X_n . A Matsuki partner of $G_0 = \mathrm{SO}(1, 2n)$ is the complex Lie group $K \subset G$ having Lie algebra

$$\mathfrak{k} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & B \\ 0 & C & -A^t \end{pmatrix}; A \in \mathbb{C}^{n \times n}, B, C \in \mathfrak{so}(n, \mathbb{C}) \right\} \simeq \mathfrak{so}(2n, \mathbb{C}).$$

The group $K \simeq \mathrm{SO}(2n, \mathbb{C})$ has three orbits in X_n : the open one consists of all isotropic subspaces of \mathbb{C}^{2n+1} that are not contained in $\{0\} \times \mathbb{C}^{2n}$, while the set of isotropic n -dimensional complex subspaces of $\{0\} \times \mathbb{C}^{2n}$ has two connected components C_0 and C_1 , both homogeneous under K , see [GH78, Proposition, p. 735]. Remark that $C_0 \cup C_1$ is homogeneous under $\widehat{K} \simeq \mathrm{O}(2n, \mathbb{C})$. Theorem A.1 and Proposition 3.2 show that (C_0, C_1) is a Schottky pair in X_n . This can also be seen directly as follows.

First we claim that the pair (C_0, C_1) is movable. Let $g \in G$ and note that $g(C_0)$ and $g(C_1)$ are the connected components of the space of isotropic n -dimensional subspaces of $g(\{0\} \times \mathbb{C}^{2n})$. If $C_0, C_1, g(C_0)$ and $g(C_1)$ are not pairwise disjoint, then there exists an isotropic n -dimensional subspace of $W_g := (\{0\} \times \mathbb{C}^{2n}) \cap g(\{0\} \times \mathbb{C}^{2n})$. However, for a generic choice of g we have $\dim W_g = 2n - 1$ and $W_g \cap W_g^\perp = \{0\}$. Therefore the dimension of an isotropic subspace of W_g is at most $n - 1$, which proves the claim.

We consider now the analogous situation in \mathbb{C}^{2n+2} with linear coordinates (u, z, w) where $u = (u_1, u_2) \in \mathbb{C}^2$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $w = (w_1, \dots, w_n) \in \mathbb{C}^n$. Let q be the quadratic form given $q(u, z, w) = u_1^2 + u_2^2 + 2\langle z, w \rangle$. The Lie algebra of its isometry group $\widehat{G} \simeq \mathrm{SO}(2n+2, \mathbb{C})$ is given by

$$\widehat{\mathfrak{g}} = \left\{ \begin{pmatrix} D & E & F \\ -F^t & A & B \\ -E^t & C & -A^t \end{pmatrix} ; D \in \mathfrak{so}(2, \mathbb{C}), E, F \in \mathbb{C}^{2 \times n}, A \in \mathbb{C}^{n \times n}, B, C \in \mathfrak{so}(n, \mathbb{C}) \right\}.$$

Take the $(n+1)$ -dimensional isotropic subspace $\widehat{V}_0 := \{(u, iu, z, 0); u \in \mathbb{C}, z \in \mathbb{C}^n\}$ and set $\widehat{X}_n := \widehat{G} \cdot \widehat{V}_0$. Note that \widehat{X}_n is one of the two connected components of the manifold of isotropic $(n+1)$ -dimensional complex subspaces of \mathbb{C}^{2n+2} and $\dim_{\mathbb{C}} \widehat{X}_n = \frac{n(n+1)}{2}$.

Let G be the subgroup of \widehat{G} having Lie algebra

$$\mathfrak{g} = \left\{ \begin{pmatrix} D & E & F \\ -F^t & A & B \\ -E^t & C & -A^t \end{pmatrix} \in \widehat{\mathfrak{g}} ; D = 0, e_{1j} = f_{1j} = 0 \text{ for all } 1 \leq j \leq n \right\} \simeq \mathfrak{so}(2n+1, \mathbb{C}).$$

Calculating the dimension of the isotropy group $G_{\widehat{V}_0}$, we see that $G \cdot \widehat{V}_0$ is open in \widehat{X}_n . Since $G_{\widehat{V}_0}$ is parabolic, it follows that G acts in fact transitively on \widehat{X}_n . It turns out that $G_{\widehat{V}_0} = Q_{\Pi \setminus \{\alpha_n\}}$, which implies that \widehat{X}_n is G -equivariantly isomorphic to the set X_n of isotropic n -dimensional complex subspaces of \mathbb{C}^{2n+1} , compare Remark A.4. In other words, the group of holomorphic automorphisms of X_n is isomorphic to $\widehat{G} \simeq \mathrm{SO}(2n+2, \mathbb{C})$ (cf. [Oni62] or [Ste82]).

Thus the manifolds X_n and \widehat{X}_n are the same. The pair of disjoint compact K -orbits (C_0, C_1) constructed above in X_n for $K \simeq \mathrm{SO}(2n, \mathbb{C})$, can be seen in \widehat{X}_n as the pair of compact orbits of the subgroup (also called) K of $\mathrm{SO}(2n+2, \mathbb{C})$ with Lie algebra

$$\mathfrak{k} = \left\{ \begin{pmatrix} D & E & F \\ -F^t & A & B \\ -E^t & C & -A^t \end{pmatrix} \in \widehat{\mathfrak{g}} ; D = 0, E = F = 0 \right\}.$$

This pair is movable already under the smaller group G as proved before. In \widehat{X}_n we can now see explicitly the \mathbb{C}^* -action which makes (C_0, C_1) a Schottky pair. The one-dimensional complex Lie group has Lie algebra

$$\mathcal{Z}_{\widehat{\mathfrak{g}}}(\mathfrak{k}) = \left\{ \begin{pmatrix} D & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; D \in \mathfrak{so}(2, \mathbb{C}) \right\} \simeq \mathbb{C},$$

and it is given (as indicated in the preceding formula) by the centralizer of K in \widehat{G} .

Remark 4.5. These Schottky pairs are closely related to Schottky groups acting by conformal automorphisms on the sphere S^{2n} as follows, see [CNS13, Chapter 10]. Consider the following real $\mathrm{SO}(2n+1)$ -equivariant fibration

$$\begin{array}{ccc} X_n = \widehat{X}_n = \mathrm{SO}(2n+2)/\mathrm{U}(n+1) & \xrightarrow{\simeq} & \mathrm{SO}(2n+1)/\mathrm{U}(n) \\ & & \downarrow \pi \\ & & \mathrm{SO}(2n+1)/\mathrm{SO}(2n) = S^{2n}. \end{array}$$

This is the so called *twistor fibration* of X_n . The fibers of π are complex manifolds isomorphic to X_{n-1} , but the foliation is *not* holomorphic. The Möbius group $\mathrm{Möb}_+(S^{2n}) = \mathrm{Conf}_+(S^{2n})$

$\simeq \mathrm{SO}(1, 2n+1)$ of conformal orientation-preserving diffeomorphisms of S^{2n} lifts to a holomorphic action of $\mathrm{SO}(1, 2n+1) \subset \mathrm{SO}(2n+2, \mathbb{C}) \simeq \mathrm{Aut}(X)$, see [CNS13, p.235–239]. This implies that *real* Schottky group actions on the manifold S^{2n} induce via lifting by π holomorphic Schottky group actions on X_n .

Let us be more precise. The natural action of the group $L := \mathbb{R}^{>0}$ as homotheties S^{2n} has two fix points, p and q , say. Lifting with π and complexifying the lifted group to $L^{\mathbb{C}} := \pi^*(L)^{\mathbb{C}} \simeq \mathbb{C}^*$, gives us a Schottky pair

$$(4.1) \quad (C_0 = \pi^{-1}(p), C_1 = \pi^{-1}(q))$$

in X_n with the property that C_0 and C_1 are biholomorphic to X_{n-1} . This Schottky pair is movable since pairs of points are movable in S^{2n} under the action of $\mathrm{SO}(1, 2n+1)$. Furthermore, one can take the function φ to be the pull-back of the standard $\mathrm{SO}(2n)$ -invariant exhaustion function on S^{2n} .

In closing we show that $\mathrm{Aut}(X_n)$ acts transitively on the set of Schottky pairs in X_n . Since two subgroups of $\mathrm{SO}(2n+2, \mathbb{C})$ isomorphic to $\mathrm{SO}(2n, \mathbb{C})$ are conjugate, for two copies C_0, C_1 of X_{n-1} in X_n there is $h \in \mathrm{SO}(2n+2, \mathbb{C})$ such that $h(C_0) = C_1$. It is then easy to see that the variety of all X_{n-1} 's in X_n is isomorphic to the even-dimensional quadric $Q_{2n} = \mathrm{SO}(2n+2, \mathbb{C})/P$. Now let (C_0, C_1) be the Schottky pair from (4.1) and denote by $L^{\mathbb{C}} \subset \mathrm{Aut}(X_n)$ the group isomorphic to \mathbb{C}^* corresponding to it. In order to prove transitivity on Schottky pairs, it is sufficient to prove that for a Schottky component C'_1 such that $C'_1 \cap C_0 = \emptyset$ there is an automorphism $g \in P$ such that $g(C_1) = C'_1$, i.e., g stabilizes C_0 and maps C_1 to C'_1 . In other words, one has to show that P acts transitively on the set of X_{n-1} 's in X_n which are disjoint from C_0 . It is well known that P has an open orbit in Q_{2n} . The isotropy group in $\mathrm{SO}(1, 2n+1)$ of $p \in S^{2n}$ acts transitively on $S^{2n} \setminus \{q\}$. This implies directly that $P \cdot C_1$ is the open P -orbit in Q_{2n} . Furthermore, since $C'_1 \cap C_0 = \emptyset$, for every arbitrarily small open neighborhood U of C_1 in X_n there is an element $g_u \in L^{\mathbb{C}} \subset P$ such that $g_u(C'_1) \subset U$. Thus $g_u(C'_1)$ is in a small open neighborhood of C_1 contained in the open orbit of P in the irreducible cycle space component isomorphic to Q_{2n} . The claim is proved.

4.5. Schottky groups acting on Q_{2n-1} . Let us discuss an example of Schottky group actions that are not directly related to minimal hypersurface orbits. For $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ let

$$X = Q_{2n-1} = \{[u : z : w] \in \mathbb{P}_{2n}; u^2 + 2\langle z, w \rangle = 0\}.$$

On X we define $\varphi[u : z : w] = \frac{\|w\|^2}{\|z\|^2 + \|w\|^2}$ and $g_\lambda[u : z : w] := [u : \lambda^{-1}z : \lambda w]$. One verifies directly that this yields a Schottky pair in X . This Schottky pair is movable by an argument similar to the one given in Subsection 4.3.

Remark 4.6. The map $p: Q_{2n-1} \rightarrow \mathbb{P}_{2n-1}$ given by $p[u : z : w] = [z : w]$ is a $2 : 1$ covering with branch locus $\{\langle z, w \rangle = 0\} \simeq Q_{2n-2}$. This covering is equivariant with respect to $G = \mathrm{SO}(2n, \mathbb{C})$.

Let us show that $\mathrm{SO}(2n+1, \mathbb{C})$ acts transitively on the set of Schottky pairs in $X = Q_{2n-1}$. If (C'_0, C'_1) is any Schottky pair in X , then there is $g \in \mathrm{SO}(2n+1, \mathbb{C})$ such that $g(C'_0) = C_0 := \{(0, z, 0); z \in \mathbb{C}^n\}$. Then $g(C'_1)$ is an n -dimensional isotropic subspace of \mathbb{C}^{2n+1} complementary to C_0 . In fact, $g(C'_1)$ must be complementary to $\{(u, z, 0); u \in \mathbb{C}, z \in \mathbb{C}^n\}$, for $(u, z, 0) \in g(C'_1)$ implies $u^2 = 0$. Now we may finish the proof in the same way as in the case of even-dimensional quadrics.

5. EXTENSION OF COHOMOLOGY GROUPS AND q -COMPLETENESS

5.1. Some extension theorems. In this subsection we collect some technical tools which allow us to study meromorphic functions, differential forms and cohomology groups of the Schottky quotients Q_Γ .

Theorem 5.1 ([Sch61]). *Let X be a d -dimensional complex manifold, $A \subset X$ a closed analytic subset of pure dimension $m - 1$, and \mathcal{F} a locally free sheaf on X . Then, for every $0 \leq k \leq d - m - 1$, the restriction map*

$$H^k(X, \mathcal{F}) \rightarrow H^k(X \setminus A, \mathcal{F})$$

is bijective.

Following Andreotti and Grauert we say that a complex manifold M of dimension d is q -complete if M admits a smooth exhaustion function ρ whose Levi form has at least $d - q + 1$ strictly positive eigenvalues at each point of M . Under this convention Stein manifolds are precisely the 1-complete manifolds.

Theorem 5.2 ([AG62, Théorème 15]). *Let Ω be a q -complete complex manifold of dimension d with exhaustion function ρ and \mathcal{F} be a locally free sheaf on Ω . Then, for every $0 \leq k \leq d - q - 1$, the restriction map*

$$H^k(\Omega, \mathcal{F}) \rightarrow H^k(\Omega \setminus \{\rho < \varepsilon\}, \mathcal{F})$$

is bijective.

We also need an extension theorem for meromorphic functions and a q -completeness criterion.

Theorem 5.3 ([MP09]). *Let Ω be a q -complete complex manifold of dimension d with $q \leq d - 1$ and $K \subset \Omega$ a compact subset. Then every meromorphic function $f \in \mathcal{M}(\Omega \setminus K)$ extends uniquely as a meromorphic function to Ω .*

Proposition 5.4 ([AN71, Proposition 8]). *Let $X \subset \mathbb{P}_n$ be a projective manifold and let s_1, \dots, s_q be holomorphic sections in the hyperplane line bundle of \mathbb{P}_n . Then $\Omega := X \setminus \{s_1 = \dots = s_q = 0\}$ is q -complete.*

5.2. Cohomology groups of Q_Γ . Our goal here is to determine certain cohomology groups of Q_Γ and for this we prove the following lemma.

Lemma 5.5. *Let X be either \mathbb{P}_{2n+1} or Q_{2n} or Q_{2n+1} and let (C_0, C_1) be one of the movable Schottky pairs in X described in Section 4. Then $X \setminus C_0$ is $(n + 1)$ -complete. For $n \geq 4$, the complex manifold $X_n \setminus C$, $C := \pi^{-1}(q)$, $q \in S^{2n}$, see Subsection 4.4, is $(d - 3)$ -complete with $d := \dim_{\mathbb{C}} X_n$.*

Proof. We apply Proposition 5.4 to each of the first three cases separately.

For $C_0 = \{[z : 0] \in \mathbb{P}_{2n+1}; z \in \mathbb{P}_n\}$ in $X = \mathbb{P}_{2n+1}$ the criterion of Andreotti and Norguet immediately yields that $X \setminus C_0$ is $(n + 1)$ -complete.

If $C_0 = \{[z : 0] \in Q_{2n}; z \in \mathbb{P}_n\}$ in $X = Q_{2n}$, then $X \setminus C_0$ is again $(n + 1)$ -complete.

Consider $C_0 = \{[0 : 0 : w] \in Q_{2n+1}; w \in \mathbb{P}_n\}$ in $X = Q_{2n+1} = \{[u : z : w]; u^2 + 2\langle z, w \rangle = 0\}$. Then we have $C_0 = \{z_1 = \dots = z_{n+1} = 0\}$, so that $X \setminus C_0$ is $(n + 1)$ -complete.

In order to prove the claim for X_4 we use the spinor embedding $X_4 \hookrightarrow \mathbb{P}_{15}$. The image of X_4 in \mathbb{P}_{15} is the closure of the set of homogeneous coordinates

$$[1 : x_{12} : \dots : x_{45} : y_1 : \dots : y_5] \in \mathbb{P}_{15}$$

where x_{kl} , $1 \leq k < l \leq 5$ are the upper triangular entries of a skew-symmetric matrix $A \in \mathbb{C}^{5 \times 5}$ and y_1, \dots, y_5 are the one-codimensional Pfaffians of A .

In [IM04, p. 291] one finds explicit equations for the image of X_4 in \mathbb{P}_{15} . Using these, it is not hard to see that the Schottky pair (C_0, C_1) in \mathbb{P}_{15} with

$$\begin{aligned} C_0 &= \{x_{12} = \cdots = x_{15} = y_2 = \cdots = y_5 = 0\} \\ C_1 &= \{u = x_{23} = \cdots = x_{45} = y_1 = 0\} \end{aligned}$$

intersects X_4 in the two connected components of $\text{IGr}_3(\mathbb{C}^6) \subset X_4$. Consequently, we have an explicit formula for Nori's function $\varphi \in \mathcal{C}^\infty(\mathbb{P}_{15})$ as well as for the tangent space of $\{\varphi = 1/2\} \cap X_4$ at $p = [1 : 1 : 0 : \cdots : 0]$. This allows us to see by a direct calculation that the Levi form of the restriction $\varphi|_{X_4}$ at p has four strictly positive eigenvalues. Since the generic fiber of $\varphi|_{X_4}$ coincides with the hypersurface orbit of $K_0 = \text{SO}(2) \times \text{SO}(8)$, we conclude that the Levi form of every K_0 -invariant exhaustion function on $X_4 \setminus (C_0 \cap X_4)$ has at least four strictly positive eigenvalues at each point. Hence, $X_4 \setminus (C_0 \cap X_4)$ is 7-complete, as claimed.

For $n \geq 5$, we consider the twistor fibration

$$\begin{array}{c} X_n = \text{SO}(2n+1)/\text{U}(n) \\ \pi \downarrow \\ S^{2n} = \text{SO}(2n+1)/\text{SO}(2n), \end{array}$$

and let p, q the two fixed points of the action of the isotropy group $\text{SO}(2n)$ on S^{2n} . Then $S^{2n} \setminus \{q\}$ is conformally isomorphic to $\mathbb{R}^n = \{x = (x_1, \dots, x_n)\}$. We identify the point p with the origin in \mathbb{R}^n and define the function $\rho(x) := \sum x_i^2$ on \mathbb{R}^n . The functions ρ and $\tilde{\rho} := \rho \circ \pi$ are invariant under the left action of $\text{SO}(2n)$ on $S^{2n} \setminus \{q\}$ and $X_n \setminus C$ and are exhaustion functions.

It is easy to check that there is a commutative diagram

$$\begin{array}{ccc} X_4 = \text{SO}(9)/\text{U}(4) & \xrightarrow{\iota_1} & \text{SO}(2n+1)/\text{U}(n) \\ \pi|_{X_4} \downarrow & & \downarrow \pi \\ S^8 = \text{SO}(9)/\text{SO}(8) & \xrightarrow{\iota_2} & \text{SO}(2n+1)/\text{SO}(2n) = S^{2n}, \end{array}$$

such that X_4 is equivariantly and holomorphically embedded in X_n for $n \geq 4$. As we have seen above, the Levi form of the restriction of $\tilde{\rho}$ to $X_4 \setminus (X_4 \cap C_0)$ has four strictly positive eigenvalues everywhere. Since $\tilde{\rho}$ is $\text{SO}(2n)$ -invariant, the same is true for $\tilde{\rho}$ on X_n . Therefore $X_n \setminus C$ is $(d-3)$ -complete. \square

Combining Lemma 5.5 with the extension theorems of Andreotti-Grauert and Scheja yields some information about cohomology groups of the Schottky quotient manifolds Q_Γ . For the following proposition it is crucial that the neighborhoods of the Schottky pair (C_0, C_1) are defined via the K_0 -invariant function φ , compare the important assumption.

Proposition 5.6. *Let X be either \mathbb{P}_{2n+1} with $n \geq 3$ or Q_{4n+2} with $n \geq 2$ or Q_{2n+1} with $n \geq 3$ or X_n with $n \geq 4$. Let (C_0, C_1) be a movable Schottky pair in X and let Γ be an associated Schottky group of rank $r \geq 2$. Let \mathcal{F} be a locally free analytic sheaf on Q_Γ such that $\pi^*\mathcal{F}$ extends to a locally free sheaf on X with $H^p(X, \pi^*\mathcal{F}) = 0$ for $p = 1, 2$. Then, for $0 \leq k \leq 2$, we have isomorphisms*

$$H^k(Q_\Gamma, \mathcal{F}) \simeq H^k(\Gamma, H^0(X, \pi^*\mathcal{F})).$$

Moreover, $\mathcal{M}(Q_\Gamma)$ can be identified with the set of Γ -invariant rational functions on X .

Proof. In the first step we will show $H^k(\mathcal{U}_\Gamma, \pi^*\mathcal{F}) \simeq H^k(X, \pi^*\mathcal{F})$ for $0 \leq k \leq 2$. To see this, note first that the fundamental domain \mathcal{F}_Γ contains the submanifold $C = f(C_0)$ for some $f \in \text{Aut}(X)$. This follows from the fact that the neighborhoods U_j and V_j are defined by the K_0 -invariant function φ . Due to Theorem 5.1 the restriction map $H^k(\mathcal{U}_\Gamma, \pi^*\mathcal{F}) \rightarrow H^k(\mathcal{U}_\Gamma \setminus C, \pi^*\mathcal{F})$ is bijective for $0 \leq k \leq 2$. Since $\mathcal{U}_\Gamma \setminus C$ is a domain in the q -complete manifold $\Omega = X \setminus C$ with compact complement, an application of Theorem 5.2 and Lemma 5.5 yields that the restriction map $H^k(X \setminus C, \pi^*\mathcal{F}) \rightarrow H^k(\mathcal{U}_\Gamma \setminus C, \pi^*\mathcal{F})$ is also bijective for $0 \leq k \leq 2$. Another application of Theorem 5.1 gives the result.

Consequently we get $H^1(\mathcal{U}_\Gamma, \pi^*\mathcal{F}) = 0 = H^2(\mathcal{U}_\Gamma, \pi^*\mathcal{F})$. This allows us to apply [Mum08, Appendix to §2, formula (c)] to obtain

$$H^k(Q_\Gamma, \mathcal{F}) \simeq H^k(\Gamma, H^0(\mathcal{U}_\Gamma, \pi^*\mathcal{F})) = H^k(\Gamma, H^0(X, \pi^*\mathcal{F}))$$

for $k = 0, 1, 2$. □

Remark 5.7. Let X be any homogeneous rational manifold, let \mathcal{F} be either the structure sheaf \mathcal{O} or the tangent sheaf Θ . Then the Bott-Borel-Weil theorem shows $H^k(X, \mathcal{F}) = 0$ for all $k \geq 1$.

Remark 5.8. For $X = \mathbb{P}_3$ or $X = Q_3$ our method only yields $H^0(Q_\Gamma, \mathcal{F}) \simeq H^0(X, \pi^*\mathcal{F})^\Gamma$. In addition, for $X = \mathbb{P}_5, Q_5, Q_6$ we have $H^1(Q_\Gamma, \mathcal{F}) \simeq H^1(\Gamma, H^0(\pi^*\mathcal{F}))$.

6. GEOMETRIC PROPERTIES AND DEFORMATIONS OF SCHOTTKY QUOTIENTS

In this section we apply Proposition 5.6 in order to describe analytic and geometric invariants as well as the deformation theory of Schottky quotient manifolds. In the whole section X will denote a homogeneous rational manifold admitting a movable Schottky pair (C_0, C_1) and Γ an associated Schottky group of rank $r \geq 2$ with quotient $Q_\Gamma = \mathcal{U}_\Gamma/\Gamma$.

6.1. Analytic and geometric invariants. The following proposition was shown in [Lár98] for $X = \mathbb{P}_n$ under an additional assumption on the Hausdorff dimension of $X \setminus \mathcal{U}_\Gamma$.

Proposition 6.1. *The quotient manifold Q_Γ is rationally connected and has Kodaira dimension $-\infty$. If $\text{codim } C_0 \geq 2$, then Q_Γ is not Kähler.*

Proof. The first claim follows again from the fact that we define the open neighborhoods U_j and V_j via a K_0 -invariant function $\varphi: X \rightarrow [0, 1]$. In this situation we find enough rational curves in the fundamental domain \mathcal{F}_Γ so that we can connect any two points by a chain of rational curves.

In order to show $\text{kod}(Q_\Gamma) = -\infty$ we apply Proposition 5.6 to the canonical sheaf \mathcal{K}_{Q_Γ} . Then the claim follows from $H^0(X, \mathcal{K}_X^{\otimes m}) = 0$ for every $m \geq 1$ since X is rational.

The last claim is a consequence of the fact that the fundamental group of a compact Kähler manifold cannot be free. This can be seen as follows, compare [ABCKT96, Example 1.19]. Since the rank of the free fundamental group of a compact complex manifold Y coincides with the first Betti number of Y , we see that no compact Kähler manifold can have free fundamental of odd rank. However, any free group of rank r contains normal subgroups of any finite index k which are free of rank $k(r-1) + 1$ due to the Nielsen-Schreier theorem. If a compact Kähler manifold Y had free fundamental group of even rank, we could choose a normal subgroup of even index k and thus would obtain a finite covering of Y having free fundamental group of odd rank $k(r-1) + 1$, a contradiction to the previous observation. This proves the last claim since \mathcal{U}_Γ is simply-connected if $\text{codim } C_0 \geq 2$. □

Next we give a criterion for the algebraic dimension of Q_Γ to be zero.

Theorem 6.2. *The algebraic dimension $a(Q_\Gamma)$ coincides with the codimension of a generic H -orbit in X where H denotes the Zariski closure of Γ in $\text{Aut}(X)$. In particular, $a(Q_\Gamma) = 0$ if and only if H has an open orbit in X .*

Proof. Due to Theorem 5.3 and Lemma 5.5 every meromorphic function on Q_Γ is induced by a Γ -invariant rational function f on X . Consequently, f must be invariant under the Zariski closure H of Γ in $\text{Aut}(X)$. It follows from Rosenlicht's theorem [Ro56, Theorem 2] that the field of H -invariant rational functions on X has transcendence degree equal to the codimension of a generic H -orbit in X . \square

Remark 6.3. It is not difficult to produce examples of Schottky groups Γ acting on \mathbb{P}_{2n+1} such that $a(Q_\Gamma) = 0$: choose $r = 2n + 1$ pairwise disjoint Schottky pairs such that in some point of \mathbb{P}_{2n+1} the corresponding \mathbb{C}^* -orbits meet transversally.

It is shown in [Lár98, Proposition 2.1] that Nori's Schottky groups acting on $X = \mathbb{P}_3$ yield quotient manifolds of algebraic dimension zero, provided that the Hausdorff dimension of their limit set is sufficiently small. It can be deduced from Theorem 6.2 that this assumption on the Hausdorff dimension is superfluous. In [CNS13, Proposition 9.3.12] and [SV03, Proposition 3.5] the same result is claimed to hold for Schottky groups acting on $X = \mathbb{P}_{2n+1}$. This, however, is not correct as the following example shows.

Example 6.4. Let us fix two integers $k \geq 1$ and $n \geq 2k + 1$. Applying Nori's construction to $X = \mathbb{P}(\mathbb{C}^{(2k) \times n}) \simeq \mathbb{P}_{2kn-1}$ gives the Schottky pair

$$\begin{aligned} C_0 &:= \{[Z] \in X; z_{ij} = 0 \text{ for all } k+1 \leq i \leq 2k\} \simeq \mathbb{P}_{kn-1} \text{ and} \\ C_1 &:= \{[Z] \in X; z_{ij} = 0 \text{ for all } 1 \leq i \leq k\} \simeq \mathbb{P}_{kn-1}. \end{aligned}$$

The corresponding \mathbb{C}^* -action is given by $g_\lambda \in \text{Aut}(X)$,

$$g_\lambda[Z] = g_\lambda \left[\begin{pmatrix} Z_0 \\ Z_1 \end{pmatrix} \right] := \left[\begin{pmatrix} \lambda^{-1} Z_0 \\ \lambda Z_1 \end{pmatrix} \right]$$

where $Z_0, Z_1 \in \mathbb{C}^{k \times n}$. Let the group $H = \text{SL}(2k, \mathbb{C})$ act on X by left multiplication. We have $g_\lambda \in H$ for all $\lambda \in \mathbb{C}^*$. Moreover, a direct calculation shows that the Schottky pair (C_0, C_1) can be moved by elements of H . Consequently, there are Schottky groups Γ acting on X with $\Gamma \subset H$.

Due to the First Fundamental Theorem (see e.g. [Pro07, p. 387]) for H the invariant ring $\mathbb{C}[\mathbb{C}^{(2k) \times n}]^H$ is generated by the $\binom{n}{2k}$ homogeneous $(2k) \times (2k)$ minors of A . Since $n \geq 2k + 1$, there are non-constant H -invariant rational functions on X . Thus the algebraic dimension of Q_Γ is bounded from below by $2k(n - 2k)$ for every Schottky group $\Gamma \subset H$.

Note that C_0 and C_1 are contained in the H -invariant subvariety $\overline{Y_k}$ where $Y_k := \{[Z] \in X; \text{rk}(Z) = k\}$. Therefore we can view $\overline{Y_k}$ as another projective variety (of dimension $k(k + n) - 1$) which admits actions of Schottky groups. For $k \geq 2$ the projective variety $\overline{Y_k}$ is singular and C_0 and C_1 meet its singular set $\overline{Y_k} \setminus Y_k$. In contrast, $Y_1 = \overline{Y_1}$ is H -equivariantly isomorphic to $\mathbb{P}_1 \times \mathbb{P}_{n-1}$ where $H = \text{SL}(2, \mathbb{C})$ acts on $\mathbb{P}_1 \times \mathbb{P}_{n-1}$ by $h \cdot ([v], [z]) := ([hv], [z])$. Hence, the Schottky groups acting on Y_1 are obtained as products of Schottky groups acting on \mathbb{P}_1 and the trivial action on \mathbb{P}_{n-1} .

In the next example we study in detail a fiber of the algebraic reduction map obtained in the setting of Example 6.4.

Example 6.5. Take now in the setting of the previous example $n = 2k$. In this case the natural right action of $\text{SL}(2k, \mathbb{C})$ on $X = \mathbb{P}(\mathbb{C}^{(2k) \times (2k)}) \simeq \mathbb{P}_{4k^2-1}$ commutes with the left action and

therefore descends to an action on Q_Γ which is constructed as the quotient of the *left* action of Γ . In particular, the limit set $X \setminus \mathcal{U}_\Gamma$ must be contained in the complement of the open $\mathrm{SL}(2k, \mathbb{C})$ -orbit. Hence Q_Γ is an almost homogeneous $\mathrm{SL}(2k, \mathbb{C})$ -manifold with open orbit $\Gamma \backslash \mathrm{SL}(2k, \mathbb{C})$.

This gives in the case $k = 1$ interesting $\mathrm{SL}(2, \mathbb{C})$ -equivariant compactifications of $\Gamma \backslash \mathrm{SL}(2, \mathbb{C})$ that to the best of our knowledge are new. We describe them here in a little more detail.

Consider $X = \mathbb{P}_3 \simeq \mathbb{P}(\mathbb{C}^{2 \times 2})$ as an almost-homogeneous complex manifold under the action of the complex Lie group $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$, given by left and right matrix multiplication. Then $X = \mathrm{PSL}(2, \mathbb{C}) \cup D$, where D is the 1-codimensional orbit isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$. As above construct now a (left) Schottky group action on X of $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$. Restricted to D this action is trivial on one \mathbb{P}_1 -factor and just a classical Schottky action on the other. Therefore the quotient manifold Q_Γ is an equivariant holomorphic compactification of $\Gamma \backslash \mathrm{SL}(2, \mathbb{C})$ by a hypersurface $D' \simeq R \times \mathbb{P}_1$, where R is the compact Riemann surface associated to the classical Schottky action of Γ on \mathbb{P}_1 . By a theorem of Maskit ([Ma67]), it follows that for every finitely generated free, discrete and loxodromic subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ the above construction gives such an equivariant compactification.

A similar construction is also possible for the case of Schottky group actions on quadrics as well as on isotropic Grassmannians as the following two examples show.

Example 6.6. As in Subsection 4.3 we equip \mathbb{C}^{4k} with the symmetric bilinear form $b(z, w) := z^t S w$ where $S = \begin{pmatrix} 0 & I_{2k} \\ I_{2k} & 0 \end{pmatrix}$. Let H be the group of linear isometries of b and note that $H \simeq \mathrm{SO}(4k, \mathbb{C})$. On $\mathbb{C}^{4k \times m}$ for $m \geq 1$ we define a symmetric bilinear form B by the formula $B(Z, W) := \mathrm{Tr}(Z^t S W)$. Then any two different columns of $Z \in \mathbb{C}^{4k \times m}$ are orthogonal with respect to B and, when restricted to one column, B coincides with b . The group $H \times \mathrm{SO}(m, \mathbb{C})$ acts on $\mathbb{C}^{4k \times m}$ by left and right multiplication and this action leaves B invariant. Consequently, $H \times \mathrm{SO}(m, \mathbb{C})$ acts on the $(4k - 2)$ -dimensional quadric

$$X := \{[Z] \in \mathbb{P}(\mathbb{C}^{4k \times m}); \mathrm{Tr}(Z^t S Z) = 0\}$$

in $\mathbb{P}(\mathbb{C}^{4k \times m}) \simeq \mathbb{P}_{4km-1}$. In contrast with the previous example, $\mathrm{SO}(m, \mathbb{C})$ does not have an open orbit in X .

An argument analogous to the one given in Subsection 4.3 shows that the Schottky pair

$$\begin{aligned} C_0 &= \left\{ \left[\begin{pmatrix} Z_0 \\ 0 \end{pmatrix} \right] \in X; Z_0 \in \mathbb{C}^{2k \times m} \right\} \simeq \mathbb{P}_{2km-1} \\ C_1 &= \left\{ \left[\begin{pmatrix} 0 \\ Z_1 \end{pmatrix} \right] \in X; Z_1 \in \mathbb{C}^{2k \times m} \right\} \simeq \mathbb{P}_{2km-1} \end{aligned}$$

in X is movable by elements of the group H . Therefore there are Schottky groups Γ acting on X with Zariski closure $\overline{\Gamma}$ contained in H .

Due to the First Fundamental Theorem for $\mathrm{SO}(4k, \mathbb{C})$ the algebra of H -invariant polynomials on $\mathbb{C}^{4k \times m}$ is generated by

$$p_{ij}(Z) = b(z_i, z_j) \quad \text{for } m \geq 1, 1 \leq i \leq j \leq m,$$

where z_1, \dots, z_m are the columns of $Z \in \mathbb{C}^{4k \times m}$, and by

$$d_{i_1 \dots i_{4k}}(Z) = \det(z_{i_1} \cdots z_{i_{4k}}) \quad \text{for } m \geq 4k, 1 \leq i_1 < i_2 < \cdots < i_{4k} \leq m.$$

Hence, for each $m \geq 2$ and every Schottky group $\Gamma \subset H$ the quotient manifold Q_Γ has positive algebraic dimension.

Concretely, suppose that $k = 1$ and $m = 2$ and let Γ be Zariski dense in H . Then we have

$$X = \{[(z_1 \ z_2)] \in \mathbb{P}(\mathbb{C}^{4 \times 2}); b(z_1, z_1) + b(z_2, z_2) = 0\}.$$

The algebraic reduction map of Q_Γ is induced by the rational mapping

$$X \dashrightarrow \mathbb{P}_2, [Z] \mapsto [b(z_1, z_1) : b(z_1, z_2) : b(z_2, z_2)],$$

whose image is contained in $\{[x_0 : x_1 : x_2] \in \mathbb{P}_2; x_0 = -x_2\} \simeq \mathbb{P}_1$. Note that this reduction map is $\mathrm{SO}(2, \mathbb{C})$ -equivariant.

Example 6.7. Here we construct Schottky groups acting on $X_n = \mathrm{IGr}_n(\mathbb{C}^{2n+1})$ with sufficiently small Zariski closures in $\mathrm{SO}(2n+2, \mathbb{C})$ in order to have Γ -invariant non-constant meromorphic functions. Then these functions induce meromorphic functions on the associated quotient manifolds which will be hence of strictly positive algebraic dimension, compare Theorem 6.2.

To this end we use the twistor fibration, see Remark 4.5. Let $M := S^m$ be a round sphere in S^{2n} . Its stabilizer is a subgroup of $\mathrm{SO}(1, 2n+1)$ isomorphic to $\mathrm{Möb}_+(S^m) \simeq \mathrm{SO}(1, m+1)$. Construct a Schottky group action on S^{2n} by allowing the pairs of points (p, q) to move only in M . Then the lifted Schottky group has a Zariski closure H contained in a subgroup of $\mathrm{SO}(2n+2, \mathbb{C})$ isomorphic to $\mathrm{SO}(m+2, \mathbb{C})$. Finally, if m is sufficiently small, there are H -invariant meromorphic functions on X_n and the quotient manifold has strictly positive algebraic dimension.

6.2. The Picard group of Q_Γ . Let X be a homogeneous rational manifold verifying the hypotheses of Proposition 5.6. Applying this proposition to the structure sheaf $\mathcal{F} = \mathcal{O}$ we obtain $H^1(Q_\Gamma, \mathcal{O}) \simeq H^1(\Gamma, \mathbb{C}) \simeq \mathrm{Hom}(\Gamma_{\mathrm{ab}}, \mathbb{C}) \simeq \mathbb{C}^r$, see [HilSt97, p. 193], as well as $H^2(Q_\Gamma, \mathcal{O}) \simeq H^2(\Gamma, \mathbb{C}) = 0$, see [HilSt97, Corollary VI.5.6]. Hence, the long exact cohomology sequence associated with the exponential sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \rightarrow \mathcal{O}^* \rightarrow 0$ yields

$$\begin{array}{ccccccc} H^1(Q_\Gamma, \mathbb{Z}) & \hookrightarrow & H^1(Q_\Gamma, \mathcal{O}) & \rightarrow & H^1(Q_\Gamma, \mathcal{O}^*) & \rightarrow & H^2(Q_\Gamma, \mathbb{Z}) \rightarrow 0. \\ \simeq \mathbb{Z}^r & & \simeq \mathbb{C}^r & & & & \end{array}$$

In order to obtain the Picard group $H^1(Q_\Gamma, \mathcal{O}^*)$ we have to determine $H^2(Q_\Gamma, \mathbb{Z})$.

Remark 6.8. The subgroup $H^1(Q_\Gamma, \mathcal{O})/H^1(Q_\Gamma, \mathbb{Z}) \simeq (\mathbb{C}^*)^r$ of the Picard group of Q_Γ consists of the topologically trivial line bundles on Q_Γ , given by representations of Γ in \mathbb{C}^* .

In a first step we will determine $H_2(\mathcal{U}_\Gamma) := H_2(\mathcal{U}_\Gamma, \mathbb{Z})$. For this, we note that \mathcal{U}_Γ is the increasing union of the open sets

$$\Omega_l := X \setminus \left(\bigcup_{j=1}^r \bigcup_{\substack{\gamma \in \Gamma_l \\ \gamma_{j_l} \neq \gamma_j}} \gamma(\overline{U}_j) \right) \cup \left(\bigcup_{j=1}^r \bigcup_{\substack{\gamma \in \Gamma_l \\ \gamma_{j_l} \neq \gamma_j^{-1}}} \gamma(\overline{V}_j) \right)$$

where Γ_l denotes the set of all reduced words of length $l \geq 1$ in Γ . Since Ω_l is homotopy equivalent to $X \setminus C$ where C is the union of $N_l = 2r(2r-1)^{l-1}$ pairwise disjoint copies of C_0 , we have $H_k(\Omega_l) \cong H_k(X \setminus C)$. Let U be a tubular neighborhood of C having N_l connected components homeomorphic to $C_0 \times B$ where B is the unit ball in $\mathbb{R}^{\dim_{\mathbb{R}} X - \dim_{\mathbb{R}} C_0}$. Then the Mayer-Vietoris sequence of the open cover $X = U \cup (X \setminus C)$ with $U \cap (X \setminus C) = U \setminus C$ reads

$$\begin{aligned} \cdots \rightarrow H_{k+1}(X) &\rightarrow H_k(U \setminus C) \rightarrow H_k(U) \oplus H_k(X \setminus C) \rightarrow \\ &\rightarrow H_k(X) \rightarrow H_{k-1}(U \setminus C) \rightarrow \cdots \rightarrow H_0(X) \rightarrow 0. \end{aligned}$$

Note that U is homotopy equivalent to the disjoint union of N_l copies of C_0 while $U \setminus C$ is homotopy equivalent to the disjoint union of N_l copies of $C_0 \times S^{\dim_{\mathbb{R}} X - \dim_{\mathbb{R}} C_0 - 1}$ where S^d is the unit sphere in \mathbb{R}^{d+1} . Since C_0 is homogeneous rational, its homology $H_*(C_0)$ is free

Abelian. Therefore the Künneth formula yields

$$H_k(C_0 \times S^d) \simeq \bigoplus_{j=0}^k H_j(C_0) \otimes H_{k-j}(S^d).$$

Consider the case $k = 2$ and suppose that $d \geq 3$. Then we have $H_2(C_0 \times S^d) \simeq H_2(C_0)$. Furthermore, the Mayer-Vietoris sequence starting at $H_3(X) = 0$ looks like

$$0 \rightarrow H_2(U \setminus C) \xrightarrow{\simeq H_2(C_0)^{N_l}} H_2(U) \oplus H_2(X \setminus C) \xrightarrow{\simeq H_2(C_0)^{N_l}} H_2(X) \rightarrow H_1(U \setminus C) = 0.$$

From this we obtain $H_2(\Omega_l) \simeq H_2(X \setminus C) \simeq H_2(X)$ for all $l \geq 1$. Since every singular chain in \mathcal{U}_Γ lies in Ω_l for some $l \geq 1$, we conclude $H_2(\mathcal{U}_\Gamma) \simeq H_2(X)$.

In order to deduce $H_2(Q_\Gamma, \mathbb{Z})$, we will use in the second step the Cartan-Leray spectral sequence. More precisely, there exists a first quadrant spectral sequence of homology type with

$$E_2^{p,q} \simeq H_p(\Gamma, H_q(\mathcal{U}_\Gamma))$$

and strongly converging to $H_*(Q_\Gamma)$, see [McCl01, Theorem 8^{bis}.9].

Since Γ is free, we have $E_2^{p,q} = 0$ for $p \geq 2$, see [HilSt97, Corollary VI.5.6]. Since the differential d_2 is of bidegree $(-2, 1)$ we obtain

$$\begin{array}{ccccccc} & & * & & * & & 0 & & 0 \\ & & \swarrow & & \swarrow & & \swarrow & & \swarrow \\ & & & & 0 & & & & 0 \\ H_0(\Gamma, H_2(X)) & & H_1(\Gamma, H_2(X)) & & 0 & & 0 & & 0 \\ & & \swarrow & & \swarrow & & \swarrow & & \swarrow \\ & & & & 0 & & 0 & & 0 \\ 0 & & 0 & & 0 & & 0 & & 0 \\ & & \swarrow & & \swarrow & & \swarrow & & \swarrow \\ H_0(\Gamma, \mathbb{Z}) & & H_1(\Gamma, \mathbb{Z}) & & 0 & & 0 & & 0 \end{array}$$

Consequently, this spectral sequence collapses at $r = 2$ and we have $E_2^{p,q} = E_\infty^{p,q}$ for all p, q . In other words, there is an increasing filtration F^* on $H_*(Q_\Gamma)$ such that

$$\begin{aligned} 0 &= E_2^{2,0} \simeq F^2 H_2(Q_\Gamma) / F^1 H_2(Q_\Gamma) \\ 0 &= E_2^{1,1} \simeq F^1 H_2(Q_\Gamma) / F^0 H_2(Q_\Gamma) \\ E_2^{0,2} &\simeq F^0 H_2(Q_\Gamma). \end{aligned}$$

Since Γ is contained in a connected complex Lie group, the induced action of Γ on $H_q(\mathcal{U}_\Gamma)$ is trivial. Hence, we have $E_2^{0,2} \simeq H_0(\Gamma, H_2(X)) \simeq H_2(X)$, see [HilSt97, Proposition VI.3.1].

In summary, we have shown $H_2(Q_\Gamma) \simeq H_2(X)$. It follows from [CS09, Theorem 3.2.20] that $H_2(X) \simeq \mathbb{Z}$ for every homogeneous rational manifold verifying the hypotheses of Proposition 5.6. From this we obtain the following.

Theorem 6.9. *Let X be a homogeneous rational manifold verifying the hypotheses of Proposition 5.6. Let Γ be a Schottky group of rank r acting on X with associated quotient $\pi: \mathcal{U}_\Gamma \rightarrow Q_\Gamma$. Then the Picard group $H^1(Q_\Gamma, \mathcal{O}^*)$ of Q_Γ is isomorphic to $(\mathbb{C}^*)^r \times \mathbb{Z}$.*

6.3. Deformation theory of Q_Γ . It is possible to embed a Schottky quotient manifold Q_Γ into a complex analytic family in the following way. Fix a movable Schottky pair (C_0, C_1) in X and automorphisms $f_2, \dots, f_r \in \text{Aut}(X)$ such that $C_0, C_1, f_2(C_0), f_2(C_1), \dots, f_r(C_0), f_r(C_1)$ are pairwise disjoint. Then, as above, we may choose elements $\lambda_1, \dots, \lambda_r \in \mathbb{C}^*$ with $|\lambda_j|^2 = \frac{1-\varepsilon_j}{\varepsilon_j} > 1$. Let $\mathcal{D} \subset \mathbb{C}^r$ be the domain of all possible such $\lambda := (\lambda_1, \dots, \lambda_r)$. Set $f_1 := \text{id}_X$ and write $\Gamma(\lambda)$ for the Schottky group generated by $\gamma_j := f_j \circ g_{\lambda_j} \circ f_j^{-1}$ for $1 \leq j \leq r$. Let $F_r = \langle s_1, \dots, s_r \rangle$ be the abstract free group of rank r . We have an action of the free group F_r of rank r on $X \times \mathcal{D}$ given by the formula

$$s_j \cdot (x, \lambda) := (f_j \circ g_{\lambda_j} \circ f_j^{-1}(x), \lambda).$$

Let $U_j(\lambda)$ and $V_j(\lambda)$ be the open neighborhoods of $f_j(C_0)$ and $f_j(C_1)$, respectively, defined for the $\Gamma(\lambda)$ -action on X . Then set $\widehat{U}_j := \{(x, \lambda) \in X \times \mathcal{D}; x \in U_j(\lambda)\}$ and similarly \widehat{V}_j for $1 \leq j \leq r$. In the same way we define $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{U}}$ in $X \times \mathcal{D}$.

Proposition 6.10. *The free group F_r acts freely and properly on $\widehat{\mathcal{U}}$ so that we obtain the commutative diagram*

$$\begin{array}{ccc} \widehat{\mathcal{U}} & \longrightarrow & \widehat{\mathcal{U}}/F_r \\ \downarrow & & \downarrow \pi \\ \mathcal{D} & \longrightarrow & \mathcal{D}. \end{array}$$

The map $\pi: \widehat{\mathcal{U}}/F_r \rightarrow \mathcal{D}$ is a complex analytic family in the sense of Kodaira with fibers $Q_{\Gamma(\lambda)}$. In particular, all Schottky quotient manifolds are diffeomorphic.

Proof. Since we have $s_j(\widehat{U}_j) = (X \times \mathcal{D}) \setminus \widehat{V}_j$ for all $1 \leq j \leq r$, we may literally copy the proof of the corresponding fact without the parameters λ . \square

Proposition 6.11. *Two Schottky quotient manifolds Q_Γ and $Q_{\Gamma'}$ are biholomorphic if and only if Γ and Γ' are conjugate in $\text{Aut}(X)$.*

Proof. Suppose that there exists a biholomorphic map $f: Q_\Gamma \rightarrow Q_{\Gamma'}$. Then there exist a biholomorphic map $F: \mathcal{U}_\Gamma \rightarrow \mathcal{U}_{\Gamma'}$ as well as a group homomorphism $\varphi: \Gamma \rightarrow \Gamma'$ such that $F \circ \gamma = \varphi(\gamma) \circ F$ for all $\gamma \in \Gamma$. Since X is projective, F is given by finitely many meromorphic functions f_1, \dots, f_N on \mathcal{U}_Γ . Due to Theorem 5.3 and Lemma 5.5 we may thus extend F as a meromorphic map to X . It is not hard to show that this extended map is biholomorphic, see [Iva92], hence an element of $\text{Aut}(X)$. Consequently, Γ and Γ' are conjugate in $\text{Aut}(X)$. \square

In the rest of this subsection we assume in addition that X verifies the hypotheses of Proposition 5.6. In this case, Proposition 5.6 applied to the tangent sheaf Θ gives $H^k(Q_\Gamma, \Theta) \simeq H^k(\Gamma, \mathfrak{g})$ for $0 \leq k \leq 2$ where Γ acts on \mathfrak{g} via the adjoint representation. Explicitly, we get $H^0(\Gamma, \mathfrak{g}) \simeq \mathfrak{g}^\Gamma$ where \mathfrak{g}^Γ denotes the subspace of Γ -fixed points. Let H be the Zariski closure of Γ in G . Then we have $\mathfrak{g}^\Gamma = \mathfrak{g}^H$ which in turn coincides with the centralizer $\mathcal{Z}_{\mathfrak{g}}(\mathfrak{h})$ if H is connected. Moreover, the group of biholomorphic automorphisms $\text{Aut}(Q_\Gamma)$ is a complex Lie group with Lie algebra \mathfrak{g}^H . As noted in [HilSt97, p. 195], $H^1(\Gamma, \mathfrak{g})$ is isomorphic to the quotient of $\text{Hom}_\Gamma(I\Gamma, \mathfrak{g})$, where $I\Gamma$ denotes the augmentation ideal of Γ , by the submodule of homomorphisms of the form $\varphi_\xi: \gamma - e \mapsto \text{Ad}(\gamma)\xi - \xi$. Under the identification $\text{Hom}_\Gamma(I\Gamma, \mathfrak{g}) \simeq \mathfrak{g}^r$ this submodule corresponds to the image of the map $\psi: \mathfrak{g} \rightarrow \mathfrak{g}^r$ given by the formula

$$\psi(\xi) = (\text{Ad}(\gamma_1)\xi - \xi, \dots, \text{Ad}(\gamma_r)\xi - \xi),$$

i.e., $H^1(\Gamma, \mathfrak{g}) \simeq \mathfrak{g}^r / \psi(\mathfrak{g})$. Note that the kernel of ψ is \mathfrak{g}^Γ . Finally, $H^2(\Gamma, \mathfrak{g}) = 0$, see [HilSt97, Corollary VI.5.6]. In summary, we have the following.

Theorem 6.12. *Suppose that X verifies the hypotheses of Proposition 5.6. The Kuranishi space of versal deformations of Q_Γ is smooth at Q_Γ and of complex dimension $(r-1)\dim \mathfrak{g} + \dim \mathfrak{g}^\Gamma$. Moreover, the automorphism group $\text{Aut}(Q_\Gamma)$ admits as Lie algebra \mathfrak{g}^Γ .*

- Remark 6.13.* (a) If X is either \mathbb{P}_5 or Q_5 or Q_6 , then we can still compute the dimension of the Kuranishi space. However, we do not know whether the Kuranishi space is smooth or not.
- (b) In [CNS13, Theorem 9.3.17] (see also [SV03]), the authors claim that the Kuranishi space is of dimension $(r-1)\dim \mathfrak{g}$. But in general the above mapping ψ is *not* injective, i.e., it is possible that Q_Γ has strictly positive dimensional automorphism group or is even almost homogeneous, see our examples 6.4 and 6.5.

APPENDIX A. MINIMAL ORBITS OF HYPERSURFACE TYPE

Let G be a simply-connected semisimple complex Lie group, let Q be a parabolic subgroup of G , and let G_0 be a non-compact simple real form of G . We say that two triplets (G, G_0, Q) and $(G, \tilde{G}_0, \tilde{Q})$ are equivalent if there exist $g_1, g_2 \in G$ such that $\tilde{G}_0 = g_1 G_0 g_1^{-1}$ and $\tilde{Q} = g_2 Q g_2^{-1}$. In this appendix we outline the classification (up to equivalence) of all triplets (G, G_0, Q) such that the minimal G_0 -orbit is a real hypersurface in $X = G/Q$.

Throughout we write $\sigma: \mathfrak{g} \rightarrow \mathfrak{g}$ for conjugation with respect to \mathfrak{g}_0 . Let $\theta: \mathfrak{g} \rightarrow \mathfrak{g}$ be a Cartan involution that commutes with σ . Then we have the corresponding Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$. The analytic subgroup K_0 of G_0 having Lie algebra \mathfrak{k}_0 is a maximal compact subgroup of G_0 .

The following theorem summarizes the outcome of the appendix.

Theorem A.1. *Up to equivalence, the homogeneous rational manifolds $X = G/Q$ and the real forms G_0 having a compact hypersurface orbit in X are the following:*

- (1) $G_0 = \text{SU}(p, q)$ acting on $X = \mathbb{P}_{p+q-1}$;
- (2) $G_0 = \text{Sp}(p, q)$ acting on $X = \mathbb{P}_{2(p+q)-1}$;
- (3) $G_0 = \text{SU}(1, n)$ acting on $X = \text{Gr}_k(\mathbb{C}^{n+1})$;
- (4) $G_0 = \text{SO}^*(2n)$ acting on $X = Q_{2n-2}$;
- (5) $G_0 = \text{SO}(1, 2n)$ acting on $X = \text{IGr}_n(\mathbb{C}^{2n+1})$;
- (6) $G_0 = \text{SO}(2, 2n)$ acting on $X = \text{IGr}_{n+1}(\mathbb{C}^{2n+2})^0$.

A.1. Root-theoretic description of the minimal G_0 -orbit in $X = G/Q$. Let \mathfrak{a}_0 be a maximal Abelian subspace of \mathfrak{p}_0 and let

$$\mathfrak{g}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \bigoplus_{\lambda \in \Lambda} (\mathfrak{g}_0)_\lambda$$

be the corresponding restricted root space decomposition of \mathfrak{g}_0 where $\mathfrak{m}_0 = \mathcal{Z}_{\mathfrak{k}_0}(\mathfrak{a}_0)$ and where $\Lambda = \Lambda(\mathfrak{g}_0, \mathfrak{a}_0) \subset \mathfrak{a}_0^* \setminus \{0\}$ is the restricted root system. Choosing a system Λ^+ of positive restricted roots we obtain the nilpotent subalgebra $\mathfrak{n}_0 := \bigoplus_{\lambda \in \Lambda^+} \mathfrak{g}_\lambda$. From this we get the Iwasawa decomposition $G_0 = K_0 A_0 N_0$ where A_0 and N_0 are the analytic subgroups of G_0 having Lie algebras \mathfrak{a}_0 and \mathfrak{n}_0 , respectively.

Let \mathfrak{t}_0 be a maximal torus in \mathfrak{m}_0 . Then $\mathfrak{h}_0 := \mathfrak{t}_0 \oplus \mathfrak{a}_0$ is a maximally non-compact Cartan subalgebra of \mathfrak{g}_0 . Let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

be the root space decomposition of \mathfrak{g} with respect to the Cartan subalgebra $\mathfrak{h} := \mathfrak{h}_0^\mathbb{C}$ with root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}_\mathbb{R}^* \setminus \{0\}$ where $\mathfrak{h}_\mathbb{R} := i\mathfrak{t}_0 \oplus \mathfrak{a}_0$. Let $R: \mathfrak{h}_\mathbb{R}^* \rightarrow \mathfrak{a}_0^*$ be the restriction operator and let $\Delta_i := \{\alpha \in \Delta; R(\alpha) = 0\}$ be the set of *imaginary* roots.

Remark A.2. We have $\mathfrak{m}_0^{\mathbb{C}} = \mathfrak{t}_0^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta_i} \mathfrak{g}_{\alpha}$, i.e., Δ_i is the root system of $\mathfrak{m}_0^{\mathbb{C}}$ with respect to its Cartan subalgebra $\mathfrak{t}_0^{\mathbb{C}}$.

Let us define a system Δ^+ of positive roots with respect to the lexicographic ordering given by a basis of $\mathfrak{h}_{\mathbb{R}}$ whose first r elements form a basis of \mathfrak{a}_0 . Then, for every $\alpha \in \Delta \setminus \Delta_i$, we have $\alpha \in \Delta^+$ if and only if $R(\alpha) \in \Lambda^+$ and $R(\Delta^+ \setminus \Delta_i) = \Lambda^+$, see [Vin94, p.156].

Since the anti-involution σ stabilizes $\mathfrak{h}_{\mathbb{R}}$, we obtain an induced involution on $\mathfrak{h}_{\mathbb{R}}^*$ which we denote again by σ . One checks directly that σ leaves Δ invariant and that $\Delta_i = \{\alpha \in \Delta; \sigma(\alpha) = -\alpha\}$. A root $\alpha \in \Delta$ is called *real* if $\sigma(\alpha) = \alpha$, and Δ_r is the set of real roots. Since $R(\alpha) = R(\sigma(\alpha))$ for all $\alpha \in \Delta$, we get

$$\sigma(\Delta^+ \setminus \Delta_i) = \Delta^+ \setminus \Delta_i.$$

In other words, Δ^+ is a σ -order in the terminology of [Ara62].

Before we can state the main result of this subsection, we have to review the description of parabolic subalgebras of \mathfrak{g} in terms of the root system Δ . Recall that a root $\alpha \in \Delta^+$ is called *simple* if it cannot be written as the sum of two positive roots. Let $\Pi \subset \Delta^+$ be the subset of simple roots. The elements of Π form a basis of $\mathfrak{h}_{\mathbb{R}}^*$ and every positive root can be uniquely written as a linear combination of simple roots with non-negative integer coefficients.

For an arbitrary subset Γ of Π we set $\Gamma^r := \langle \Gamma \rangle_{\mathbb{Z}} \cap \Delta$ and $\Gamma^n := \Delta^+ \setminus \Gamma^r$. Then

$$\mathfrak{q}_{\Gamma} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Gamma^r} \mathfrak{g}_{\alpha} \oplus \bigoplus_{\alpha \in \Gamma^n} \mathfrak{g}_{\alpha}$$

is a parabolic subalgebra of \mathfrak{g} . The subalgebra $\bigoplus_{\alpha \in \Gamma^n} \mathfrak{g}_{\alpha}$ is the nilradical of \mathfrak{q}_{Γ} , while the reductive subalgebra $\mathfrak{h} \oplus \bigoplus_{\alpha \in \Gamma^r} \mathfrak{g}_{\alpha}$ is a Levi subalgebra of \mathfrak{q}_{Γ} . Let Q_{Γ} be the analytic subgroup of G having Lie algebra \mathfrak{q}_{Γ} . Then Q_{Γ} is a parabolic subgroup of G and every parabolic subgroup of G is conjugate to Q_{Γ} for a suitable choice of $\Gamma \subset \Pi$.

After replacing the triplet (G, G_0, Q) by an equivalent one we may assume that $G_0 \cdot eQ$ is compact in G/Q . Due to [Wol69, Lemma 3.1] this means that there exist a maximally non-compact Cartan subalgebra \mathfrak{h}_0 of \mathfrak{g}_0 and a σ -order Δ^+ of $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ such that $Q = Q_{\Gamma}$ for a suitable subset $\Gamma \subset \Pi$. By [Wol69, Theorem 2.12] the real codimension of $G_0 \cdot eQ$ in X is given by $|\Gamma^n \cap \sigma(\Gamma^n)|$. Therefore the minimal G_0 -orbit is a hypersurface if and only if $\Gamma^n \cap \sigma(\Gamma^n) = \{\alpha_0\}$ for some $\alpha_0 \in \Delta^+$.

Suppose that the minimal G_0 -orbit in $X = G/Q$ is a hypersurface. Then we have $\sigma(\alpha_0) = \alpha_0$, i.e., Δ_r^+ cannot be empty. This implies that the Lie algebra \mathfrak{g} must be simple, too, and that there are at least two conjugacy classes of Cartan subalgebras in \mathfrak{g}_0 . Furthermore, it is not hard to see that, if \mathfrak{g}_0 is a split real form, then $G_0 \simeq \mathrm{SL}(2, \mathbb{R})$ and $X \simeq \mathbb{P}_1$.

The strategy of the classification is as follows. For every complex simple Lie algebra \mathfrak{g} and for every real form \mathfrak{g}_0 we determine explicitly the corresponding involution σ of $\mathfrak{h}_{\mathbb{R}}^*$ and a σ -order Δ^+ of $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$. Then we enumerate all subsets $\Gamma \subset \Pi \subset \Delta^+$ such that $\Gamma^n \cap \sigma(\Gamma^n) = \{\alpha_0\}$. This procedure will result in the list given in the beginning of Section 4.

In closing let us note that, if the compact G_0 -orbit in $X = G/Q$ is a hypersurface, then X is K -spherical. Since the triplets (G, G_0, Q) such that $X = G/Q$ is K -spherical are classified in [HONO13, Table 2], the number of possibilities of Γ that have to be checked is further reduced.

The necessary information about root systems and Satake diagrams can be found in [Hel01, Chapter X.3.3 and Table VI].

A.2. The series A_n . Let $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$ with $n \geq 1$. The root system of \mathfrak{g} is given by

$$\Delta = \{\pm(e_k - e_l); 1 \leq k < l \leq n+1\}$$

where (e_1, \dots, e_{n+1}) is the standard basis of \mathbb{R}^{n+1} and Δ is contained in the hypersurface $\{x \in \mathbb{R}^{n+1}; x_1 + \dots + x_{n+1} = 0\}$. For $\Delta^+ := \{e_k - e_l; 1 \leq k < l \leq n+1\}$ we have

$$\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_n = e_n - e_{n+1}\}.$$

A direct calculation shows

$$e_k - e_l = \alpha_k + \dots + \alpha_{l-1}$$

for all $1 \leq k < l \leq n+1$.

The non-compact real forms of \mathfrak{g} are $\mathfrak{sl}(n+1, \mathbb{R})$, $\mathfrak{sl}((n+1)/2, \mathbb{H})$ if $n+1$ is even, and $\mathfrak{su}(p, q)$ with $1 \leq p \leq q$ and $p+q = n+1$. Since $\mathfrak{sl}(n, \mathbb{R})$ is a split real form and since $\mathfrak{sl}(n, \mathbb{H})$ contains only one Cartan subalgebra up to conjugation, see [Kna02, Appendix C.3], we can restrict attention to $\mathfrak{g}_0 := \mathfrak{su}(p, q)$. The real rank of \mathfrak{g}_0 is $\text{rk}_{\mathbb{R}} \mathfrak{g}_0 := \dim \mathfrak{a}_0 = p$ and the restricted root system Λ is $(\text{BC})_p$ for $p < q$ and C_p for $p = q$. The action of σ on Δ is given by

$$\sigma(e_k) = \begin{cases} -e_{n+2-k} & : 1 \leq k \leq p \text{ or } q+1 \leq k \leq n+1 \\ -e_k & : p+1 \leq k \leq q \end{cases}.$$

This follows from the fact that $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ is conjugate to the Abelian Lie algebra consisting of all matrices of the form

$$\text{diag}(it_p + s_1, it_{p-1} + s_2, \dots, it_1 + s_p, ir_1, \dots, ir_{q-p}, it_1 - s_p, \dots, it_p - s_1)$$

where $t_k, s_l, r_m \in \mathbb{R}$ such that $2(t_1 + \dots + t_p) + r_1 + \dots + r_{q-p} = 0$. One verifies directly that Δ^+ is a σ -order.

Remark A.3. Let $\Gamma_k := \Pi \setminus \{\alpha_k\}$ for $1 \leq k \leq n$. Then we have $\Gamma_k^n = \{e_1 - e_{k+1}, \dots, e_1 - e_{n+1}, \dots, e_k - e_{k+1}, \dots, e_k - e_{n+1}\}$. The cardinality of Γ_k^n is $k(n+1-k)$. The corresponding homogeneous rational manifold is $X = G/Q_{\Gamma_k} \simeq \text{Gr}_k(\mathbb{C}^{n+1})$.

Claim. If the minimal G_0 -orbit in $X = G/Q$ is a hypersurface, then Q is a maximal parabolic subgroup of G , i.e., $Q = Q_{\Gamma_k}$ for some $1 \leq k \leq n$.

Proof. Exclude the trivial case $p = q = 1$ and suppose that Q_{Γ} is not maximal, i.e., that $\Gamma \subset \Pi \setminus \{\alpha_k, \alpha_l\}$ for some $1 \leq k < l \leq n$. Then Γ^n contains

$$\begin{aligned} \Gamma_k^n \cup \Gamma_l^n = \{e_1 - e_{k+1}, \dots, e_1 - e_{n+1}, \dots, e_k - e_{k+1}, \dots, e_k - e_{n+1}, \\ e_{k+1} - e_{l+1}, \dots, e_{k+1} - e_{n+1}, \dots, e_l - e_{l+1}, \dots, e_l - e_{n+1}\}. \end{aligned}$$

If p is arbitrary and $l = n$, then Γ^n contains $e_1 - e_j$ and $e_j - e_{n+1}$ for all $k+1 \leq j \leq n$. Since we have excluded $p = q = 1$, either we have $1 = p < q$ or $2 \leq p \leq q$. In the first case $\Gamma^n \cap \sigma(\Gamma^n)$ contains $e_1 - e_{p+1}$ and $e_{p+1} - e_{n+1}$, while in the second case $\Gamma^n \cap \sigma(\Gamma^n)$ contains $e_1 - e_2$ and $e_n - e_{n+1}$. Hence, in both cases the minimal G_0 -orbit is not a hypersurface.

If $p = 1$ and $1 \leq k < l \leq n-1$, then $\Gamma^n \cap \sigma(\Gamma^n)$ contains again $e_1 - e_2$ and $e_2 - e_{n+1}$ so that the minimal G_0 -orbit is not a hypersurface.

Suppose finally that $p \geq 2$ and $1 \leq k < l \leq n-1$. Then $\Gamma^n \cap \sigma(\Gamma^n)$ contains $e_1 - e_{n+1}$ and $e_2 - e_n$, which finishes the proof of the claim. \square

Claim. The minimal orbit of $G_0 = \text{SU}(1, n)$ is a hypersurface in $X = G/Q_{\Gamma_k}$ for every $1 \leq k \leq n$.

Proof. Since $p = 1$, we have $\sigma(e_j) = -e_j$ for all $2 \leq j \leq n$. Therefore, the only roots in Γ_k^n which are not imaginary are $e_1 - e_{k+1}, \dots, e_1 - e_{n+1}$ and $e_2 - e_{n+1}, \dots, e_k - e_{n+1}$. But for $2 \leq j \leq n$ only one of the roots $e_1 - e_j$ and $\sigma(e_1 - e_j) = e_j - e_{n+1}$ can belong to Γ_k^n , which proves $\Gamma_k^n \cap \sigma(\Gamma_k^n) = \{e_1 - e_{n+1}\}$. \square

Claim. The minimal G_0 -orbit in $X = G/Q_{\Gamma_1} \simeq \mathbb{P}_n$ and $X = G/Q_{\Gamma_n} \simeq \mathbb{P}_n$ is a hypersurface for any $G_0 = \mathrm{SU}(p, q)$.

Proof. If $\Gamma = \Pi \setminus \{\alpha_1\}$, then $\Gamma^n = \{e_1 - e_2, \dots, e_1 - e_{n+1}\}$. For every $2 \leq j \leq n+1$ we have $\sigma(e_1 - e_j) = -\sigma(e_j) - e_{n+1}$ and $\sigma(e_j) = -e_1$ occurs only for $j = n+1$, which proves $\Gamma^n \cap \sigma(\Gamma^n) = \{e_1 - e_{n+1}\}$.

The case $\Gamma = \Pi \setminus \{\alpha_n\}$ can be treated similarly. \square

In order to take care of the remaining cases we show

Claim. Suppose that $p \geq 2$ and $2 \leq k \leq n-1$. Then the minimal G_0 -orbit in $X = G/Q_{\Gamma_k}$ is not a hypersurface.

Proof. Since $2 \leq k \leq n-1$, the set Γ_k^n contains the two roots $e_1 - e_{n+1}$ and $e_2 - e_n$. Moreover, due to $p \geq 2$, these roots are real, hence the claim follows. \square

In summary, we have established the first and third entry in the list given in the beginning of Section 4.

A.3. The series B_n . Let $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$. The root system of \mathfrak{g} is given by

$$\Delta = \{\pm e_k; 1 \leq k \leq n\} \cup \{\pm e_k \pm e_l; 1 \leq k < l \leq n\}$$

where (e_1, \dots, e_n) is the standard basis of \mathbb{R}^n . For $\Delta^+ := \{e_k, e_k \pm e_l; 1 \leq k < l \leq n\}$ we have

$$\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n\}.$$

A direct calculation shows

$$\begin{aligned} e_k &= \alpha_k + \dots + \alpha_n \\ e_k - e_l &= \alpha_k + \dots + \alpha_{l-1} \\ e_k + e_l &= \alpha_k + \dots + \alpha_{l-1} + 2(\alpha_l + \dots + \alpha_n) \end{aligned}$$

for all $1 \leq k < l \leq n$.

The only non-compact real forms of \mathfrak{g} are $\mathfrak{g}_0 := \mathfrak{so}(p, q)$ with $1 \leq p \leq q$ and $p+q = 2n+1$. The real rank of \mathfrak{g}_0 is $\mathrm{rk}_{\mathbb{R}} \mathfrak{g}_0 = p$ and the restricted root system Λ coincides with B_p . The action of σ on Δ is given by

$$\sigma(e_k) = \begin{cases} e_k & : 1 \leq k \leq p \\ -e_k & : p+1 \leq k \leq p + \left[\frac{q-p}{2}\right] = n \end{cases}.$$

Therefore the simple roots $\alpha_1, \dots, \alpha_{p-1}$ are real and $\alpha_{p+1}, \dots, \alpha_n$ are imaginary, while $\sigma(\alpha_p) = e_p + e_{p+1}$.

According to [HONO13, Table 2] the only $\Gamma \subset \Pi$ such that the minimal G_0 -orbit in $X = G/Q_{\Gamma}$ might be a hypersurface are the following: if $p = 1$, then $\Gamma \subset \Pi$ is arbitrary; if $p = 2$, then $\Gamma = \Pi \setminus \{\alpha_j\}$ for $1 \leq j \leq n$; if $p \geq 3$, then Γ is either $\Pi \setminus \{\alpha_1\}$ or $\Pi \setminus \{\alpha_n\}$.

Let us assume first $p \geq 2$. If $\Gamma = \Pi \setminus \{\alpha_1\}$, then Γ^n contains the real roots e_1 and $e_1 \pm e_2$ so that the minimal G_0 -orbit cannot be a hypersurface. If $2 \leq j \leq n$ and $\Gamma = \Pi \setminus \{\alpha_j\}$, then Γ^n contains the real roots e_1 and e_2 so that the minimal G_0 -orbit cannot be a hypersurface.

Assume now that $p = 1$. If Γ does not contain α_j for some $1 \leq j \leq n-1$, then Γ^n contains $e_1 \pm e_n$, so that the minimal G_0 -orbit cannot be a hypersurface. On the other hand, for $\Gamma = \Pi \setminus \{\alpha_n\}$ we have $\Gamma^n = \{e_1, \dots, e_n, e_k + e_l; 1 \leq k < l \leq n\}$, hence $\Gamma^n \cap \sigma(\Gamma^n) = \{e_1\}$. In this case the minimal G_0 -orbit is a hypersurface.

Remark A.4. Let $\Gamma = \Pi \setminus \{\alpha_n\}$. Then $G_0 = \mathrm{SO}(1, 2n)$ has a compact hypersurface orbit in $X = G/Q_{\Gamma}$. We have $\dim_{\mathbb{C}} X = |\Gamma^n| = n(n+1)/2$. Note that $X \simeq \mathrm{IGr}_n(\mathbb{C}^{2n+1})$.

A.4. The series C_n . Let $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$ with $n \geq 2$. The root system of \mathfrak{g} is given by

$$\Delta = \{2e_k; 1 \leq k \leq n\} \cup \{\pm e_k \pm e_l; 1 \leq k < l \leq n\}$$

where (e_1, \dots, e_n) is the standard basis of \mathbb{R}^n . For $\Delta^+ := \{2e_k, e_k \pm e_l; 1 \leq k < l \leq n\}$ we have

$$\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n\}.$$

A direct calculation shows that

$$\begin{aligned} 2e_k &= 2(\alpha_k + \dots + \alpha_{n-1}) + \alpha_n \\ e_k - e_l &= \alpha_k + \dots + \alpha_{l-1} \\ e_k + e_l &= \alpha_k + \dots + \alpha_{l-1} + 2(\alpha_l + \dots + \alpha_{n-1}) + \alpha_n \end{aligned}$$

for all $1 \leq k < l \leq n$.

The non-compact real forms of \mathfrak{g} are $\mathfrak{sp}(n, \mathbb{R})$ and $\mathfrak{sp}(p, q)$ with $1 \leq p \leq q$ and $p + q = n$. Since $\mathfrak{sp}(n, \mathbb{R})$ is a split real form, it is sufficient to consider $\mathfrak{g}_0 := \mathfrak{sp}(p, q)$. The real rank of \mathfrak{g}_0 is p and the restricted root system coincides with $(BC)_p$ for $p < q$ and C_p for $p = q$. The action of σ on Δ is given by

$$\sigma(e_k) = \begin{cases} e_{k+1} & : \text{if } 1 \leq k \leq 2p \text{ is odd} \\ e_{k-1} & : \text{if } 1 \leq k \leq 2p \text{ is even} \\ -e_k & : 2p+1 \leq k \leq n \end{cases}.$$

According to [HONO13, Table 2] the only $\Gamma \subset \Pi$ such that the minimal G_0 -orbit in $X = G/Q_\Gamma$ might be a hypersurface are the following: if $p = 1$, then $\Gamma = \Pi \setminus \{\alpha_k, \alpha_l\}$ for all $1 \leq k \leq l \leq n$ (with $k = l$ allowed); if $p = 2$, then $\Gamma = \Pi \setminus \{\alpha_k\}$ for all $1 \leq k \leq n$; if $p \geq 3$, then the only possibilities for Γ are $\Pi \setminus \{\alpha_k\}$ for $k = 1, 2, 3, n$ or $\Pi \setminus \{\alpha_1, \alpha_2\}$.

Let p be arbitrary. For $\Gamma = \Pi \setminus \{\alpha_1\}$ we have $\Gamma^n = \{e_1 \pm e_2, \dots, e_1 \pm e_n, 2e_1\}$ and thus $\Gamma^n \cap \sigma(\Gamma^n) = \{e_1 + e_2\}$. Hence, $G_0 = \text{Sp}(p, q)$ has a compact hypersurface orbit in $X = G/Q_\Gamma \simeq \mathbb{P}_{2n-1}$. Now suppose that Γ does not contain the root α_k for some $k \geq 2$. Then Γ^n contains $2e_1$ and $2e_2 = \sigma(2e_1)$, so that the minimal G_0 -orbit in X is not a hypersurface.

A.5. The series D_n . Let $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ with $n \geq 4$.¹ The root system of \mathfrak{g} is given by

$$\Delta = \{\pm e_k \pm e_l; 1 \leq k < l \leq n\}$$

where (e_1, \dots, e_n) is the standard basis of \mathbb{R}^n . For $\Delta^+ := \{e_k \pm e_l; 1 \leq k < l \leq n\}$ we have

$$\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\}.$$

Remark A.5. There exists an automorphism of Π that exchanges α_{n-1} and α_n . Consequently, there exists an outer automorphism of $G = \text{SO}(2n, \mathbb{C})$ that maps $Q_{\Pi \setminus \{\alpha_{n-1}\}}$ onto $Q_{\Pi \setminus \{\alpha_n\}}$ although these parabolic groups are not conjugate in G . In particular, the corresponding homogeneous rational manifolds are isomorphic. As hermitian symmetric spaces they are isomorphic to $\text{SO}(2n)/\text{U}(n)$.

A direct calculation shows that

$$\begin{aligned} e_k - e_l &= \alpha_k + \dots + \alpha_{l-1} \\ e_k + e_{n-1} &= \alpha_k + \dots + \alpha_n \text{ for all } 1 \leq k \leq n-2 \\ e_k + e_n &= \alpha_k + \dots + \alpha_{n-2} + \alpha_n \text{ for all } 1 \leq k \leq n-2 \\ e_k + e_l &= \alpha_k + \dots + \alpha_{l-1} + 2(\alpha_l + \dots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n \text{ for all } 1 \leq k < l \leq n-2. \end{aligned}$$

The non-compact real forms of \mathfrak{g} are $\mathfrak{so}^*(2n)$ and $\mathfrak{so}(p, q)$ with $1 \leq p \leq q$ and $p + q = 2n$.

¹Recall that $\mathfrak{so}(6, \mathbb{C}) \simeq \mathfrak{sl}(4, \mathbb{C})$.

Consider first $\mathfrak{g}_0 = \mathfrak{so}^*(2n)$. The real rank of \mathfrak{g}_0 is $[n/2]$ and the restricted root system is $(\text{BC})_m$ for $n = 2m + 1$ and C_m if $n = 2m$.

We start with the case that $n = 2m$ is even. The corresponding involution of Δ is induced by

$$\sigma(e_k) = \begin{cases} e_{k+1} & : 1 \leq k \leq n \text{ is odd} \\ e_{k-1} & : 1 \leq k \leq n \text{ is even} \end{cases}.$$

One verifies directly that Δ^+ is a σ -order.

For $\Gamma = \Pi \setminus \{\alpha_1\}$ we have $\Gamma^n = \{e_1 \pm e_2, \dots, e_1 \pm e_n\}$ and hence $\Gamma^n \cap \sigma(\Gamma^n) = \{e_1 + e_2\}$. Consequently, the minimal G_0 -orbit in $X = G/Q_\Gamma$ is a hypersurface. If Γ does not contain α_2 , then Γ^n contains $e_1 - e_3$ and $e_2 - e_4 = \sigma(e_1 - e_3)$, i.e., the minimal G_0 -orbit in $X = G/Q_\Gamma$ is not a hypersurface. If $n \geq 6$ and if Γ does not contain α_k for $3 \leq k \leq n$, then Γ^n contains the two real roots $e_1 + e_2$ and $e_3 + e_4$. On the other hand, for $n = 4$ and $\Gamma = \Pi \setminus \{\alpha_3\}$ we obtain $\Gamma^n \cap \sigma(\Gamma^n) = \{e_1 + e_2\}$, hence the minimal orbit of $\text{SO}^*(8)$ in $X = G/Q_\Gamma$ is a hypersurface in this case. One checks directly that in the remaining cases $\text{SO}^*(8)$ does not have a compact hypersurface orbit.

Suppose now that $n = 2m + 1 \geq 5$ is odd. In this case the involution of Δ is given by

$$\sigma(e_k) = \begin{cases} e_{k+1} & : 1 \leq k \leq n-1 \text{ is odd} \\ e_{k-1} & : 1 \leq k \leq n-1 \text{ is even} \\ -e_k & : k = n \end{cases}.$$

As above we see that for $\Gamma = \Pi \setminus \{\alpha_1\}$ we have $\Gamma^n \cap \sigma(\Gamma^n) = \{e_1 + e_2\}$, while the minimal G_0 -orbit in $X = G/Q_\Gamma$ is not a hypersurface if Γ does not contain α_k for $2 \leq k \leq n$.

In summary, the only cases in which the minimal orbit of $G_0 = \text{SO}^*(2n)$ in $X = G/Q_\Gamma$ is a hypersurface are $\Gamma = \Pi \setminus \{\alpha_1\}$ as well as $n = 4$ and $\Gamma = \Pi \setminus \{\alpha_3\}$.

Remark A.6. The exceptional case $n = 4$ is explained by $\mathfrak{so}^*(8) \simeq \mathfrak{so}(6, 2)$ which corresponds to the fact that $\text{SO}(8)/\text{U}(4)$ is isomorphic to the 3-dimensional quadric.

In the rest of this subsection we treat the case $\mathfrak{g}_0 = \mathfrak{so}(p, q)$ with $1 \leq p \leq q$ and $p + q = 2n$. The real rank of \mathfrak{g}_0 is p and the restricted root system is B_p for $p < q$ and D_p for $p = q$.

Remark A.7. The Lie algebra $\mathfrak{so}(n, n)$ is a split real form of \mathfrak{g} . The Lie algebra $\mathfrak{so}(1, 2n-1)$ contains only one conjugacy class of Cartan subalgebras, see [Kna02, Appendix C.3].

The involution of Δ is induced by

$$\sigma(e_k) = \begin{cases} e_k & : 1 \leq k \leq p \\ -e_k & : p+1 \leq k \leq p + [\frac{q-p}{2}] = n \end{cases}.$$

According to [HONO13] the minimal G_0 -orbit in $X = G/Q_\Gamma$ may be a hypersurface only in the following cases. If $p = 1$, then $\Gamma \subset \Pi$ is arbitrary; if $p = 2$, then Γ coincides with $\Pi \setminus \{\alpha_k\}$ or $\Pi \setminus \{\alpha_k, \alpha_{n-1}\}$ or $\Pi \setminus \{\alpha_k, \alpha_n\}$ for any k ; if $p \geq 3$, then the only possibilities for Γ are $\Pi \setminus \{\alpha_1\}$ or $\Pi \setminus \{\alpha_{n-1}\}$ or $\Pi \setminus \{\alpha_n\}$.

Let us begin with the case $p \geq 3$. If $\Gamma = \Pi \setminus \{\alpha_k\}$ for $k = 1, n-1, n$, then Γ^n contains the real roots $e_1 + e_2$ and $e_2 + e_3$ so that the minimal G_0 -orbit in $X = G/Q_\Gamma$ cannot be a hypersurface.

Suppose now that $p = 2$. If Γ does not contain α_k for some $1 \leq k \leq n-2$, then Γ^n contains $e_1 \pm e_n$. Since $\sigma(e_1 - e_n) = e_1 + e_n$, the minimal G_0 -orbit is not a hypersurface in this case. If $\Gamma = \Pi \setminus \{\alpha_{n-1}\}$, then we have $\Gamma^n = \{e_1 - e_n, \dots, e_{n-1} - e_n, e_k + e_l \mid (1 \leq k < l \leq n-1)\}$ and one verifies $\Gamma^n \cap \sigma(\Gamma^n) = \{e_1 + e_2\}$. Hence, the minimal G_0 -orbit in $X = G/Q_\Gamma$ is a hypersurface. For $\Gamma = \Pi \setminus \{\alpha_n\}$ we have $\Gamma^n = \{e_k + e_l : 1 \leq k < l \leq n\}$ and obtain again $\Gamma^n \cap \sigma(\Gamma^n) = \{e_1 + e_2\}$, which leads to the same conclusion as above.

Remark A.8. For $p = 1$ the above considerations show that G_0 acts transitively on $X = G/Q_\Gamma$ for $\Gamma = \Pi \setminus \{\alpha_k\}$ where $k = n - 1, n$.

In summary, the only cases in which the minimal orbit of $G_0 = \mathrm{SO}(p, q)$ in $X = G/Q_\Gamma$ is a hypersurface are $p = 2$ and $\Gamma = \Pi \setminus \{\alpha_k\}$ for $k = n - 1, n$.

A.6. The exceptional Lie algebra $\mathfrak{g} = E_6$. Combined with the general remarks in [Ara62], the Satake diagrams yield explicit formulas of the involutions corresponding to the non-split non-compact real forms of the exceptional Lie algebras E_6 , E_7 , E_8 and F_4 .

Let $\mathfrak{g} = E_6$. Identifying $\mathfrak{h}_{\mathbb{R}}^*$ with $V = \{x \in \mathbb{R}^8; x_6 = x_7 = -x_8\}$ a system of simple roots is given by

$$\Pi = \{\alpha_1 = 1/2(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), \alpha_2 = e_1 + e_2, \alpha_j = e_{j-1} - e_{j-2} (3 \leq j \leq 6)\}.$$

The Lie algebra $\mathfrak{g} = E_6$ has two non-split non-compact real forms, namely E_{II} and E_{III} .

Suppose first that $\mathfrak{g}_0 = E_{II}$. Since there is no imaginary simple root, the Satake diagram of \mathfrak{g}_0 determines directly the involution $\sigma: \Delta^+ \rightarrow \Delta^+$. More precisely, we have

$$\sigma(\alpha_1) = \alpha_6, \sigma(\alpha_3) = \alpha_5, \sigma(\alpha_2) = \alpha_2, \sigma(\alpha_4) = \alpha_4.$$

According to [HONO13] we must only check $\Pi \setminus \{\alpha_1\}$ and $\Pi \setminus \{\alpha_6\}$.

Let $\Gamma = \Pi \setminus \{\alpha_j\}$ for $j = 1, 6$. In both cases $\Gamma^n \cap \sigma(\Gamma^n)$ contains the two real roots $\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$ and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$. Hence the minimal G_0 -orbit in $X = G/Q_\Gamma$ is not a hypersurface.

Suppose now that $\mathfrak{g}_0 = E_{III}$. It can be seen from its Satake diagram that $\Pi_i = \{\alpha_3, \alpha_4, \alpha_5\}$ and that

$$\begin{aligned} \sigma(\alpha_1) &= \alpha_6 + C_{1,3}\alpha_3 + C_{1,4}\alpha_4 + C_{1,5}\alpha_5 \\ \sigma(\alpha_2) &= \alpha_2 + C_{2,3}\alpha_3 + C_{2,4}\alpha_4 + C_{2,5}\alpha_5 \\ \sigma(\alpha_6) &= \alpha_1 + C_{6,3}\alpha_3 + C_{6,4}\alpha_4 + C_{6,5}\alpha_5. \end{aligned}$$

Since σ is involutive, we obtain $C_{6,j} = C_{1,j}$ for $j = 3, 4, 5$. This gives

$$\sigma(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6) = \alpha_1 + (2C_{1,3} - 1)\alpha_3 + (2C_{1,4} - 1)\alpha_4 + (2C_{1,5} - 1)\alpha_5 + \alpha_6.$$

Comparison with the list of positive roots shows that $\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$ must be a real root, i.e., that $C_{1,3} = C_{1,4} = C_{1,5} = 1$. Similarly, the only possibilities for $\sigma(\alpha_2)$ are α_2 , $\alpha_2 + \alpha_4$, $\alpha_2 + \alpha_4 + \alpha_5$, $\alpha_2 + \alpha_3 + \alpha_4$, $\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ and $\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$. However, since we know that $\sigma(\alpha_2) - \alpha_2$ is not a root, we only have $\sigma(\alpha_2) = \alpha_2$ or $\sigma(\alpha_2) = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$. In the first case we obtain $\sigma(\alpha_2 + \alpha_4) = \alpha_2 - \alpha_4$, which contradicts the fact that Δ^+ is a σ -order. Therefore, we see that $\sigma(\alpha_2) = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$.

Let $\Gamma = \Pi \setminus \{\alpha_j\}$ for $1 \leq j \leq 6$. Then $\Gamma^n \cap \sigma(\Gamma^n)$ contains always the roots $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$ and

$$\sigma(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6) = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6.$$

Consequently, the minimal G_0 -orbit in $X = G/Q_\Gamma$ cannot be a hypersurface for any $\Gamma \subset \Pi$.

A.7. The exceptional Lie algebra $\mathfrak{g} = E_7$. Let $\mathfrak{g} = E_7$. Identifying $\mathfrak{h}_{\mathbb{R}}^*$ with $V = \{x \in \mathbb{R}^8; x_8 = -x_7\}$ a system of simple roots is given by

$$\Pi = \{\alpha_1 = 1/2(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), \alpha_2 = e_1 + e_2, \alpha_j = e_{j-1} - e_{j-2} (3 \leq j \leq 7)\}.$$

The Lie algebra $\mathfrak{g} = E_7$ has two non-split non-compact real forms, namely E_{VI} and E_{VII} .

Let $\mathfrak{g}_0 = E_{VI}$. Its Satake diagram shows $\Pi_i = \{\alpha_2, \alpha_5, \alpha_7\}$. In a first step we determine the integers $C_{k,l}$ such that

$$\sigma(\alpha_k) = \alpha_k + C_{k,2}\alpha_2 + C_{k,5}\alpha_5 + C_{k,7}\alpha_7$$

for $k = 1, 3, 4, 6$. One checks immediately that $\sigma(\alpha_1) = \alpha_1$ and $\sigma(\alpha_3) = \alpha_3$. For the remaining cases the only possibilities that respect $\sigma(\alpha_k) - \alpha_k \notin \Delta$ are

$$\sigma(\alpha_4) = \alpha_4 \text{ or } \sigma(\alpha_4) = \alpha_2 + \alpha_4 + \alpha_5$$

and

$$\sigma(\alpha_6) = \alpha_6 \text{ or } \sigma(\alpha_6) = \alpha_5 + \alpha_6 + \alpha_7.$$

Since $\alpha_4 + \alpha_5, \alpha_5 + \alpha_6 \in \Delta^+$ we obtain

$$\sigma(\alpha_4) = \alpha_2 + \alpha_4 + \alpha_5 \text{ and}$$

$$\sigma(\alpha_6) = \alpha_5 + \alpha_6 + \alpha_7.$$

According to [HONO13] the only possibility for a minimal orbit of hypersurface type is $\Gamma = \Pi \setminus \{\alpha_7\}$. Since in this case Γ^n contains the two real roots $\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$ and $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$, the minimal G_0 -orbit in $X = G/Q_\Gamma$ cannot be a hypersurface.

Let $\mathfrak{g}_0 = EVII$. Here we have $\Pi_i = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and we must determine

$$\sigma(\alpha_k) = \alpha_k + C_{k,2}\alpha_2 + C_{k,3}\alpha_3 + C_{k,4}\alpha_4 + C_{k,5}\alpha_5$$

for $k = 1, 6, 7$. One sees directly $\sigma(\alpha_7) = \alpha_7$. For the remaining cases the only possibilities that respect $\sigma(\alpha_k) - \alpha_k \notin \Delta$ are

$$\sigma(\alpha_1) = \alpha_1 \text{ or } \sigma(\alpha_1) = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$$

and

$$\sigma(\alpha_6) = \alpha_6 \text{ or } \sigma(\alpha_6) = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6.$$

Since $\alpha_1 + \alpha_3, \alpha_5 + \alpha_6 \in \Delta^+$ we obtain

$$\sigma(\alpha_4) = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 \text{ and}$$

$$\sigma(\alpha_6) = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6.$$

Let $\Gamma = \Pi \setminus \{\alpha_k\}$ for $1 \leq k \leq 7$. Then $\Gamma^n \cap \sigma(\Gamma^n)$ contains always $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$ and

$$\sigma(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7.$$

Consequently, the minimal G_0 -orbit in $X = G/Q_\Gamma$ is never a hypersurface.

A.8. The exceptional Lie algebra $\mathfrak{g} = E_8$. According to [HONO13] no real form of G can have a compact hypersurface in any G -homogeneous rational manifold.

A.9. The exceptional Lie algebra $\mathfrak{g} = F_4$. The rank of $\mathfrak{g} = F_4$ is 4 and the root system is given by

$$\Delta = \{e_k; 1 \leq k \leq 4\} \cup \{\pm e_k \pm e_l; 1 \leq k < l \leq 4\} \cup \{1/2(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}.$$

Choosing $\Delta^+ = \{e_k\} \cup \{e_k \pm e_l\} \cup \{1/2(e_1 \pm e_2 \pm e_3 \pm e_4)\}$ we obtain

$$\Pi = \{\alpha_1 = 1/2(e_1 - e_2 - e_3 - e_4), \alpha_2 = e_4, \alpha_3 = e_3 - e_4, \alpha_4 = e_2 - e_3\}.$$

The non-compact real forms of \mathfrak{g} are FI and FII . Since FI is split, we concentrate on $\mathfrak{g}_0 = FII$. According to [Ara62, p. 21] the simple roots α_2, α_3 and α_4 are imaginary while $\sigma(\alpha_1) = \alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$. Equivalently, we have $\sigma(e_1) = e_1$ and $\sigma(e_k) = -e_k$ for $2 \leq k \leq 4$. One checks that Δ^+ is a σ -order.

A direct calculation shows that the minimal G_0 -orbit in $X = G/Q_\Gamma$ is never a hypersurface.

A.10. The exceptional Lie algebra $\mathfrak{g} = G_2$. The only non-compact real form of \mathfrak{g} is split.

REFERENCES

- [Akh77] D. N. Akhiezer, *Dense orbits with two endpoints*, Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977), no. 2, 308–324, 477.
- [ABCKT96] J. Amorós, M. Burger, K. Corlette, D. Kotschick, D. Toledo, *Fundamental groups of compact Kähler manifolds*, Mathematical Surveys and Monographs **44**, American Mathematical Society, Providence, RI, 1996.
- [AG62] Aldo Andreotti and Hans Grauert, *Théorème de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90** (1962), 193–259.
- [AN71] A. Andreotti and F. Norguet, *Cycles of algebraic manifolds and $\partial\bar{\partial}$ -cohomology*, Ann. Scuola Norm. Sup. Pisa (3) **25** (1971), 59–114.
- [Ara62] Shôrô Araki, *On root systems and an infinitesimal classification of irreducible symmetric spaces*, J. Math. Osaka City Univ. **13** (1962), 1–34.
- [BL02] Ralph Bremigan and John Lorch, *Orbit duality for flag manifolds*, Manuscripta Math. **109** (2002), no. 2, 233–261.
- [Can08] Angel Cano, *Schottky groups can not act on $\mathbf{P}_{\mathbf{C}}^{2n}$ as subgroups of $\mathrm{PSL}_{2n+1}(\mathbf{C})$* , Bull. Braz. Math. Soc. (N.S.) **39** (2008), no. 4, 573–586.
- [CNS13] Angel Cano, Juan Pablo Navarrete, and José Seade, *Complex Kleinian groups*, Progress in Mathematics, vol. 303, Birkhäuser/Springer Basel AG, Basel, 2013.
- [CS09] Andreas Čap and Jan Slovák, *Parabolic geometries I. Background and general theory*, Mathematical Surveys and Monographs **154**, American Mathematical Society, Providence, RI, 2009.
- [GH78] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley-Interscience [John Wiley & Sons], New York, 1978, Pure and Applied Mathematics.
- [Hel01] Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, corrected reprint of the 1978 original, Graduate Studies in Mathematics **34**, American Mathematical Society, Providence, RI, 2001.
- [HilSt97] P. J. Hilton and U. Stammbach, *A course in homological algebra*, second ed., Graduate Texts in Mathematics 4, Springer-Verlag, New York, 1997.
- [HONO13] Xuhua He, Hiroyuki Ochiai, Kyo Nishiyama, and Yoshiki Oshima, *On orbits in double flag varieties for symmetric pairs*, Transform. Groups **18** (2013), no. 4, 1091–1136.
- [IM04] Atanas Iliev and Dimitri Markushevich, *Elliptic curves and rank-2 vector bundles on the prime Fano threefold of genus 7*, Adv. Geom. **4** (2004), no. 3, 287–318.
- [Iva92] S. Ivashkovich, *The Hartogs-type extension theorem for meromorphic maps into compact Kähler manifolds*, Invent. Math. **109** (1992), no. 1, 47–54.
- [Kna02] Anthony W. Knap, *Lie groups beyond an introduction*, second ed., Progress in Mathematics, vol. 140, Birkhäuser Boston Inc., Boston, MA, 2002.
- [Lár98] Finnur Lárusson, *Compact quotients of large domains in complex projective space*, Ann. Inst. Fourier (Grenoble) **48** (1998), no. 1, 223–246.
- [Ma67] Bernard Maskit, *A characterization of Schottky groups*, J. Analyse Math. **19** (1967), 227–230.
- [McCl01] John McCleary, *A user’s guide to spectral sequences*, second edition, Cambridge Studies in Advanced Mathematics, 58, Cambridge University Press, Cambridge, 2001.
- [MP09] Joël Merker and Egmont Porten, *The Hartogs extension theorem on $(n - 1)$ -complete complex spaces*, J. Reine Angew. Math. **637** (2009), 23–39.
- [Mum08] David Mumford, *Abelian varieties*, with appendices by C. P. Ramanujam and Yuri Manin, corrected reprint of the second (1974) ed., Tata Institute of Fundamental Research Studies in Mathematics 5, published for the Tata Institute of Fundamental Research, Bombay; by Hindustan Book Agency, New Delhi, 2008.
- [Nor86] Madhav V. Nori, *The Schottky groups in higher dimensions*, The Lefschetz centennial conference, Part I (Mexico City, 1984), Contemp. Math., vol. 58, Amer. Math. Soc., Providence, RI, 1986, pp. 195–197.
- [Oni62] A. L. Oniščik, *Inclusion relations between transitive compact transformation groups*, Trudy Moskov. Mat. Obšč. **11** (1962), 199–242.
- [Ro56] Maxwell Rosenlicht, *Some basic theorems on algebraic groups*, Amer. J. Math. **78** (1956), 401–443.
- [Pro07] Claudio Procesi, *Lie groups. An approach through invariants and representations*, Springer, New York, 2007.
- [Sch61] Günter Scheja, *Riemannsche Hebbbarkeitssätze für Cohomologieklassen*, Math. Ann. **144** (1961), 345–360.

- [Ste82] Manfred Steinsiek, *Transformation groups on homogeneous-rational manifolds*, Math. Ann. **260** (1982), no. 4, 423–435.
- [SV03] José Seade and Alberto Verjovsky, *Complex Schottky groups*, Astérisque (2003), no. 287, xx, 251–272, Geometric methods in dynamics. II.
- [Vin94] È. B. Vinberg (ed.), *Lie groups and Lie algebras, III*, Encyclopaedia of Mathematical Sciences, vol. 41, Springer-Verlag, Berlin, 1994.
- [Wol69] Joseph A. Wolf, *The action of a real semisimple group on a complex flag manifold. I. Orbit structure and holomorphic arc components*, Bull. Amer. Math. Soc. **75** (1969), 1121–1237.

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